Boundary layer flow, heat transfer and mass transfer by similarity variable solution

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Consider fluid flow approaching a semi-infinite plane surface. The fluid properties upstream of the plane are uniform velocity U, temperature T, and mole fraction ω of a chemical component that reacts at the surface. At the leading edge of the plate, a boundary layer of varying velocity, temperature and mole fraction starts to develop as the fluid flows across the surface. The flow is laminar until, at some distance down the surface, turbulent flow develops. The laminar region is considered here.

Our goal is a set of mathematical solutions that describe how velocity, temperature and mole fraction vary with position over the surface. Conservation equations can be written for conservation of mass, momentum, energy, and chemical elements. These are partial differential equations that are complex and have to be solved simultaneously to obtain general solutions.

Under conditions of the "boundary layer approximation," the equations can be simplified substantially. These assumptions for fluid flow include constant fluid density, constant pressure across the boundary layer normal to the surface, constant viscosity, negligible body forces (e.g., gravity). Orders-of-magnitude of terms are compared, and some terms are determined to be negligible and are dropped from the equations. For example, $u_x \gg u_y$, $\partial u_x / \partial y \gg \partial u_x / \partial x$, where u_x is the x-component of velocity and u_y is the y-component, and where *x* is the direction of the main fluid flow and *y* is normal to the surface. Negligible viscous dissipation is specified in the energy equation. Equimolar counter-diffusion is specified in the material balance with reaction over the surface but not in the fluid.

The simplified boundary layer equations are as follows:

Continuity Equation (conservation of total mass)

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

Equation of Motion (conservation of momentum)

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \frac{\partial^2 u_x}{\partial y^2}$$

Energy equation (conservation of energy)

$$u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

Material balance (conservation of elements)

$$u_x \frac{\partial \omega}{\partial x} + u_y \frac{\partial \omega}{\partial y} = D \frac{\partial^2 \omega}{\partial y^2}$$

The first thing to note is the similar form of the last three equations. This indicates that, if we are able to obtain a solution of the equation of motion, then we will be able to obtain solutions for the other two equations.

Boundary conditions are not shown here. Example boundary conditions are that the fluid velocity is zero at the surface, and that the temperature and mole fraction at the surface are known, or that the gradients in temperature and mole fraction at the surface are known (heat flux and material flux).

There are several approaches to solving the equations. Software packages that use numerical methods such as finite-element, finite-volume, or finite-difference methods can be used to solve the more general case of the coupled equations with varying physical properties. For the simplified equations considered here, there is the integral method and there is similarity solution method, which is used here.

Since we have specified conditions where the density, viscosity, thermal diffusivity and mass diffusivity are constant, we can solve the equation of motion separately from the other two. Then we can use that solution to solve the other two equations.

So first we will solve the equation of motion to obtain $u_x(x, y)$ and $u_y(x, y)$.

From experimental measurements, $u_x(y)$ was observed to have a similar shape at various *x* positions. The only difference was that the *y* position at which a certain value of u_x/U was obtained increased with *x*, where *U* is the fluid velocity upstream of the surface and at large y distance from the surface. Thus, we expect a solution for u_x/U which depends on some ratio of *x* and *y*, rather than *x* and *y* independently. This ratio is called the "similarity variable"

Define the "similarity variable"

$$\eta=\frac{y}{2}\sqrt{\frac{U}{\nu x}}$$

Be careful, other authors define η without the 1/2 term. At the end of this document, tables are presented that compare the notation in these notes to the notation in some representative books.

Propose a "stream function" Ψ such that Ψ satisfies the continuity equation.

$$u_x=\frac{\partial \Psi}{\partial y} \qquad u_y=-\frac{\partial \Psi}{\partial x}$$

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0$$

$$\frac{\partial^2 \Psi}{\partial x \partial y}-\frac{\partial^2 \Psi}{\partial y \partial x}=0$$

Propose that the stream function has a separable form composed of the product of a function of x and a function of η

$$\Psi(x,\eta)=H(x)F(\eta)$$

Now substitute this into the relation between Ψ and u_x

$$u_x=\frac{\partial\Psi}{\partial y}=\frac{\partial\Psi}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial y}$$

$$u_x=\frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial y}$$

$$\frac{\partial\Psi}{\partial\eta}=H(x)\frac{\partial F}{\partial\eta}=H(x)F' \quad \text{where} \quad F'=\frac{\partial F}{\partial\eta}$$

$$\frac{\partial\eta}{\partial y}=\frac{1}{2}\sqrt{\frac{U}{\nu x}}$$

Choose $H(x)=\sqrt{U\nu x}$ so that

$$u_x=\frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial y}$$

$$\frac{u_x}{U}=\frac{1}{2}F' \quad \text{EQN [1]}$$

Now substitute $\Psi(x,\eta)=H(x)F(\eta)$ into the relation between Ψ and u_y

$$-u_y=\frac{\partial\Psi}{\partial x}=\frac{\partial\Psi}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial x}$$

$$-u_y=\frac{\partial\Psi}{\partial x}=\frac{\partial\Psi}{\partial x}+\frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial x}$$

$$\frac{\partial\Psi}{\partial x}=F\frac{\partial H}{dx}=\frac{FH}{2x}$$

$$\frac{\partial\Psi}{\partial\eta}=HF'$$

$$\frac{\partial\eta}{\partial x}=-\frac{\eta}{2x}$$

$$-u_y = \frac{1}{2x} HF - \frac{\eta}{2x} HF'$$

$$\frac{u_y}{U} = \left(\frac{H}{2Ux}\right)(\eta F' - F)$$

$$\frac{u_y}{U} = \left(\frac{1}{2}\sqrt{\frac{v}{Ux}}\right)(\eta F' - F) \quad \text{EQN [2]}$$

At this stage we have relationships between u_x and F and between u_y and F in [1] and [2] above.

Now substitute these relations [1] and [2] into the Equation of Motion:

$$\frac{u_x}{U} = \frac{1}{2} F' \quad \text{EQN [1]}$$

$$\frac{u_y}{U} = \left(\frac{1}{2}\sqrt{\frac{v}{Ux}}\right)(\eta F' - F) \quad \text{EQN [2]}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = v \frac{\partial^2 u_x}{\partial y^2} \quad \text{Equation of Motion}$$

Evaluating each group from left to right,

$$u_x \frac{\partial u_x}{\partial x} = u_x \left(\frac{\partial u_x}{\partial \eta} \frac{\partial \eta}{\partial x}\right) = \left(\frac{U^2}{8x}\right)(-\eta F' F'')$$

$$u_y \frac{\partial u_x}{\partial y} = u_y \left(\frac{\partial u_x}{\partial \eta} \frac{\partial \eta}{\partial y}\right) = \left(\frac{U^2}{8x}\right)(\eta F' F'' - F F'')$$

$$\nu \frac{\partial^2 u_x}{\partial y^2} = \nu \frac{\partial}{\partial \eta}\left(\frac{\partial u_x}{\partial \eta} \frac{\partial \eta}{\partial y}\right)\frac{\partial \eta}{\partial y} = \left(\frac{U^2}{8x}\right) F'''$$

The result:

$$F''' + F F'' = 0 \quad \text{where} \quad F''' = \frac{\partial^3 F}{\partial \eta^3} \quad \text{and} \quad F'' = \frac{\partial^2 F}{\partial \eta^2}$$

This is called the **Blasius equation.**

The important thing is that, by using the similarity variable and the stream function, we have converted two partial differential equations (continuity and motion) into one ordinary differential equation (Blasius equation).

Now our task is to find $F(\eta)$ in order to satisfy $F'''+F\,F''=0$ subject to the boundary conditions:

at $\eta=0$, i.e., at the solid surface, since $u_x(\eta=0)/U=u_x(y=0)/U=0$,

$$F = 0,\ F' = 0,\ F'' = \text{unknown constant}$$

at $\eta\rightarrow\infty$, since $u_x(\eta\rightarrow\infty)/U=u_x(y\rightarrow\infty)/U=1$,

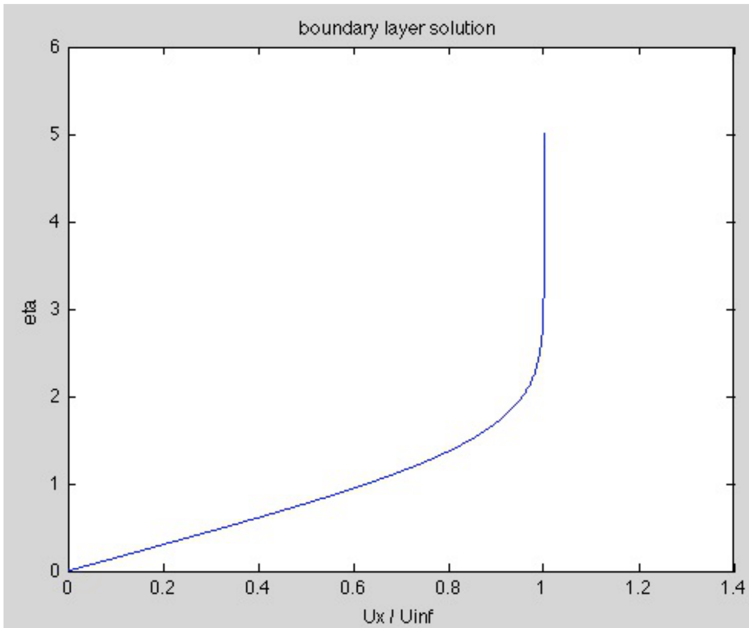
$$F' = 2$$

We have a 3rd-order ODE. An ODE of order > 1 can be reduced to multiple, coupled 1st-order ODE's. We will integrate the coupled 1st-order ODE's numerically using the iterative "shooting method."

Here is Matlab code to solve the Blasius equation:

```
% Solution of the Blasius Equation for boundary layer flow
% F''' + F * F'' = 0
% where (') specify derivative with respect to similarity variable eta
% and F' = 2 * (Ux/Uinf)
% Use of the similarity variable and the stream function
% allows the equation of motion to be converted from a PDE to an ODE
% Here solve the 3rd-order ODE by converting to three
% coupled 1st-order ODEs and solving using the iterative "shooting method"
% Our goal is to find initial conditions such that
% the asymptotic value approached at large eta is F' = 2.
% such that Ux/Uinf approaches 1 = 0.5 * (F' = 2),
% where eta = (y/2)*sqrt(Uinf/(nu*x)) and nu is kinematic viscosity

% specify initial conditions
clear
i = 1; % array index
F(i) = 0; % F = F
F1(i) = 0; % F1 = F'
F2(i)  = 1.328; % F2 = F'' <<< OUR "AIM" IN THE "SHOOTING METHOD"
% vary initial condition F2 (aim) to get final value of F1 = 2 (target)
n(i) = 0; % eta, similarity variable, eta = (y/2)*sqrt(Uinf/(nu*x))
dn = 0.001; % integration step size in n
nfinal = 5; % final value of eta to which we integrate
% where nfinal is some relatively large number, e.g., 10,
% Ux/Uinf = 0.99 at eta = 2.5 so eta = 5 should do it

% integrate using Euler's method
while n < nfinal

    % calculate derivatives
    dFdn = F1(i);
    dF1dn = F2(i);
    dF2dn = - F(i) * F2(i);

    % update estimated values
    F(i+1)  = F(i) + dFdn * dn;
    F1(i+1)  = F1(i) + dF1dn * dn;
    F2(i+1)  = F2(i) + dF2dn * dn;
    n(i+1) = n(i) + dn;
    i = i + 1;

end  % end of "while" repeat structure
```

```
% HAVE WE HIT OUR "TARGET" of F1 = F' = 2 ??
F1(i)

plot(F1/2,n)
title('boundary layer solution')
xlabel('Ux / Uinf')
ylabel('eta')
```



A value of 2 is our "target" for F1 = F' in the "shooting method." The initial value we pick for F2 = F'' is our "aim." If the final value of F1 is close to 2, then we are finished. If not, then we will need to adjust our "aim" until we hit the target. Our aim can be adjusted manually or by using a numerical algorithm that adjusts the initial value of F2 in order to minimize the square of the difference between the final value of F1 and 2.

The plot above is proportional to a plot of the velocity profile above the horizontal surface (the x-axis) as a function of y normal to the surface at some distance x from the leading edge.

The value $u_x/U=0.99$ is obtained at $\eta=2.5=(y/2)\sqrt{U/(\nu x)}$. The vertical distance y at this value of u_x/U is the "velocity boundary layer thickness," which increases with the square root of the distance x from the leading edge as long as the flow remains laminar.

Now we have a solution $u_x(x,y)$ and $u_y(x,y)$ that satisfies the continuity equation and the equation of motion.

Next, we can use this solution to solve the energy equation and the material balance to obtain temperature and mole fraction profiles.

Both of these equations can be written as follows.

$$u_x \frac{\partial \theta}{\partial x} + u_y \frac{\partial \theta}{\partial y} = \frac{\nu}{\Lambda} \frac{\partial^2 \theta}{\partial y^2}$$

where, for the energy equation,

$$\theta = \frac{T - T_{y=0}}{T_\infty - T_{y=0}} \quad \text{and} \quad \Lambda = Pr = \frac{\nu}{\alpha} \quad \text{and} \quad \alpha = \frac{k_t}{\rho C_p}$$

and for the material balance,

$$\theta = \frac{\omega - \omega_{y=0}}{\omega_\infty - \omega_{y=0}} \quad \text{and} \quad \Lambda = Sc = \frac{\nu}{D}$$

Now into this partial differential equation, substitute terms from the velocity solution procedure. As a reminder, these are,

similarity variable $\eta = \frac{y}{2}\sqrt{\frac{U}{\nu x}}$; $\frac{u_x}{U} = \frac{1}{2} F'$ EQN [1] ; $\frac{u_y}{U} = \left(\frac{1}{2}\sqrt{\frac{\nu}{U x}}\right)(\eta F' - F)$ EQN [2]

Evaluating each group from left to right,

$$u_x \frac{\partial \theta}{\partial x} = u_x \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial x} = \left(\frac{U}{4x}\right)(-\eta F')\theta' \quad \text{where} \quad \theta' = \frac{\partial \theta}{\partial \eta}$$

$$u_y \frac{\partial \theta}{\partial y} = u_y \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y} = \left(\frac{U}{4x}\right)(\eta F' - F)\theta'$$

$$\left(\frac{\nu}{\Lambda}\right) \frac{\partial^2 \theta}{\partial y^2} = \left(\frac{\nu}{\Lambda}\right) \frac{\partial}{\partial \eta}\left(\frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y}\right) \frac{\partial \eta}{\partial y} = \left(\frac{U}{4x}\right)\left(\frac{1}{\Lambda}\right)\theta''$$

The result is

$$\theta'' + \Lambda F \theta' = 0$$

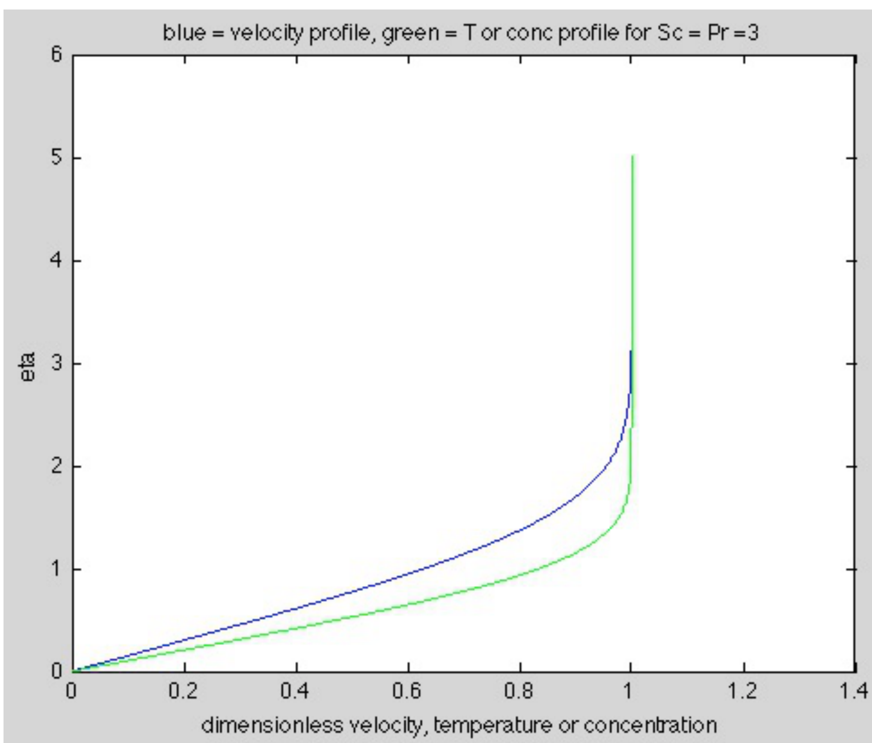
We know $F(\eta)$ from above, and we can use the shooting method to solve for $\theta(\eta)$.

Now we have solutions for the velocity, temperature, and composition profiles in the laminar boundary layer.

For $\Lambda = Pr = Sc = 1$ and for $\theta = F'/2$ we get the Blasius equation again $F''' + F F'' = 0$. This means that for *Pr* = 1, the temperature profile has the same solution as the velocity profile, and that for *Sc* = 1, the concentration profile has the same solution as the velocity profile. For other values of *Pr* and *Sc*, the temperature and concentration profiles will differ from the velocity profile.

This is a continuation of the Matlab program listing above:

```
% NOW SOLVE FOR TEMPERATURE & CONCENTRATION PROFILES
% USING SOLUTION ABOVE FOR VELOCITY PROFILE
% lambda = Sc = Pr
% theta'' + lambda * F * theta' = 0
% where theta is dimensionless T or mole fraction
% and where F is solution from above

lambda = 3; % dimensionless, Sc or Pr

% specify initial conditions
i = 1; % array index
T(i) = 0; % T = theta
T1(i) = 0.969; % T1 = theta' <<< OUR "AIM" IN THE "SHOOTING METHOD"
% T1(i) = 0.519 for lambda = 0.5; T1(i) = 0.664 for lambda = 1
% T1(i) = 0.844 for lambda = 2; T1(i) = 0.969 for lambda = 3
n(i) = 0; % eta, similarity variable, eta = (y/2)*sqrt(Uinf/(nu*x))
% MUST USE dn and nfinal FROM ABOVE SINCE USING F(n) FROM ABOVE

% integrate using Euler's method
while n(i) < nfinal
    % calculate derivatives
        dTdn = T1(i);
        dT1dn = -lambda*F(i)*T1(i);
    % update estimated values
        T(i+1)  = T(i) + dTdn * dn;
        T1(i+1)  = T1(i) + dT1dn * dn;
        n(i+1) = n(i) + dn;
        i = i + 1;
end  % end of "while" repeat structure

T(i) % HAVE WE HIT OUR "TARGET" of T = theta = 1 ??

plot(F1/2,n,'b',T,n,'g')
title(['blue = velocity profile, green = T or conc profile for Sc = Pr =',num2str(lambda)])
```



The velocity boundary layer thickness is usually defined as the distance from the surface at which u_x/U = 0.99. The temperature and concentration boundary layer thicknesses are usually defined differently:

$$\delta = \frac{1}{[\partial\theta/\partial\eta]_{\eta=0}} \quad \text{or} \quad \left[\frac{\partial\theta}{\partial\eta}\right]_{\eta=0} = \frac{1}{\delta}$$

where δ has the same dimensionless definition as η , or

$$\delta_x = \frac{1}{[\partial\theta/\partial y]_{y=0}} \quad \text{or} \quad \left[\frac{\partial\theta}{\partial y}\right]_{y=0} = \frac{1}{\delta_x}$$

where δ_x is the value of the boundary layer thickness (dimension of length) at distance x

For *Sc* and *Pr* > 1, the temperature and concentration boundary layer thicknesses are smaller than the velocity boundary layer thickness. For *Sc* and *Pr* < 1, the thermal and concentration boundary layer thicknesses are larger than the velocity boundary layer thickness.

$\frac{\delta_t}{\delta_v} \propto \frac{1}{Pr^{1/3}}$ and $\frac{\delta_c}{\delta_v} \propto \frac{1}{Sc^{1/3}}$ are approximately correct for *Pr* and *Sc* between 0.6 and 10

$$\frac{\delta_t}{\delta_v} \propto k_t^{1/3} \quad \text{and} \quad \frac{\delta_c}{\delta_v} \propto D^{1/3}$$

See other notes at this web site for "Characteristic values of some transport properties." Again, these solutions are good only in the laminar flow region for values of the Reynolds number less than about 5 x 10^5, where the characteristic length in the Reynolds number is the distance from the leading edge of the surface.

So what? What do we really want from these solutions? We want the gradients at the surface so that we can get the effect of the fluid-surface interaction.

The velocity gradient $[\partial u_x/\partial y]_{y=0}$ is proportional to the shear stress or "drag" on the surface or "wall." The temperature gradient is proportional to the heat flux at the surface. The mole fraction gradient is proportional to the molar flux at the surface.

$$\tau_{wall}(\text{N/m}^2) = \mu \left[\frac{\partial u_x}{\partial y}\right]_{y=0}$$

$$q_{wall}(\text{W/m}^2) = -k_t \left[\frac{\partial T}{\partial y}\right]_{y=0}$$

$$J_{A,wall}(\text{mol/s/m}^2) = -D\,C_{tot} \left[\frac{\partial \omega_A}{\partial y}\right]_{y=0}$$

The coefficient of friction, C_f, can be defined in terms of the velocity boundary layer thickness.

$$\tau_{wall}(\mathrm{N/m^2}) = \mu \left[\frac{\partial u_x}{\partial y} \right]_{y=0} = \frac{\rho U^2 C_f}{2}$$

$$\frac{C_f}{2} = \frac{0.332}{\sqrt{\mathrm{Re}_x}} \quad \text{where} \quad \mathrm{Re}_x = \frac{Ux}{\nu}$$

The drag force on the surface decreases with distance from the leading edge.

The local heat transfer coefficient, h_t, and mass transfer coefficient, h_c, can be defined in terms of the boundary layer thicknesses.

$$q_{wall}(\mathrm{W/m^2}) = -k_t \left[\frac{\partial T}{\partial y} \right]_{y=0} = -\left(\frac{k_t}{\delta_t} \right)(T_\infty - T_0) = -h_t (T_\infty - T_0)$$

$$J_{A,wall}(\mathrm{mol/s/m^2}) = -D C_{tot} \left[\frac{\partial \omega_A}{\partial y} \right]_{y=0} = -\left(\frac{D}{\delta_c} \right) C_{tot} (\omega_{A,\infty} - \omega_{A,0}) = -h_c C_{tot} (\omega_{A,\infty} - \omega_{A,0})$$

Since the boundary layer thicknesses increase with distance from the leading edge (proportional to the square root of the distance) in the laminar region, the local heat and mass transfer coefficients decrease with distance from the leading edge.

With larger fluid thermal conductivity k_t, the temperature boundary layer thickness is larger, the heat transfer coefficient is larger, and heat flux at the surface is larger.

$$q_{wall} \propto \frac{k_t}{\delta_t} = \frac{k_t}{k_t^{1/3}} = k_t^{2/3}$$

With larger diffusivity D, the concentration boundary layer thickness is larger, the mass transfer coefficient is larger, and molar flux at the surface is larger.

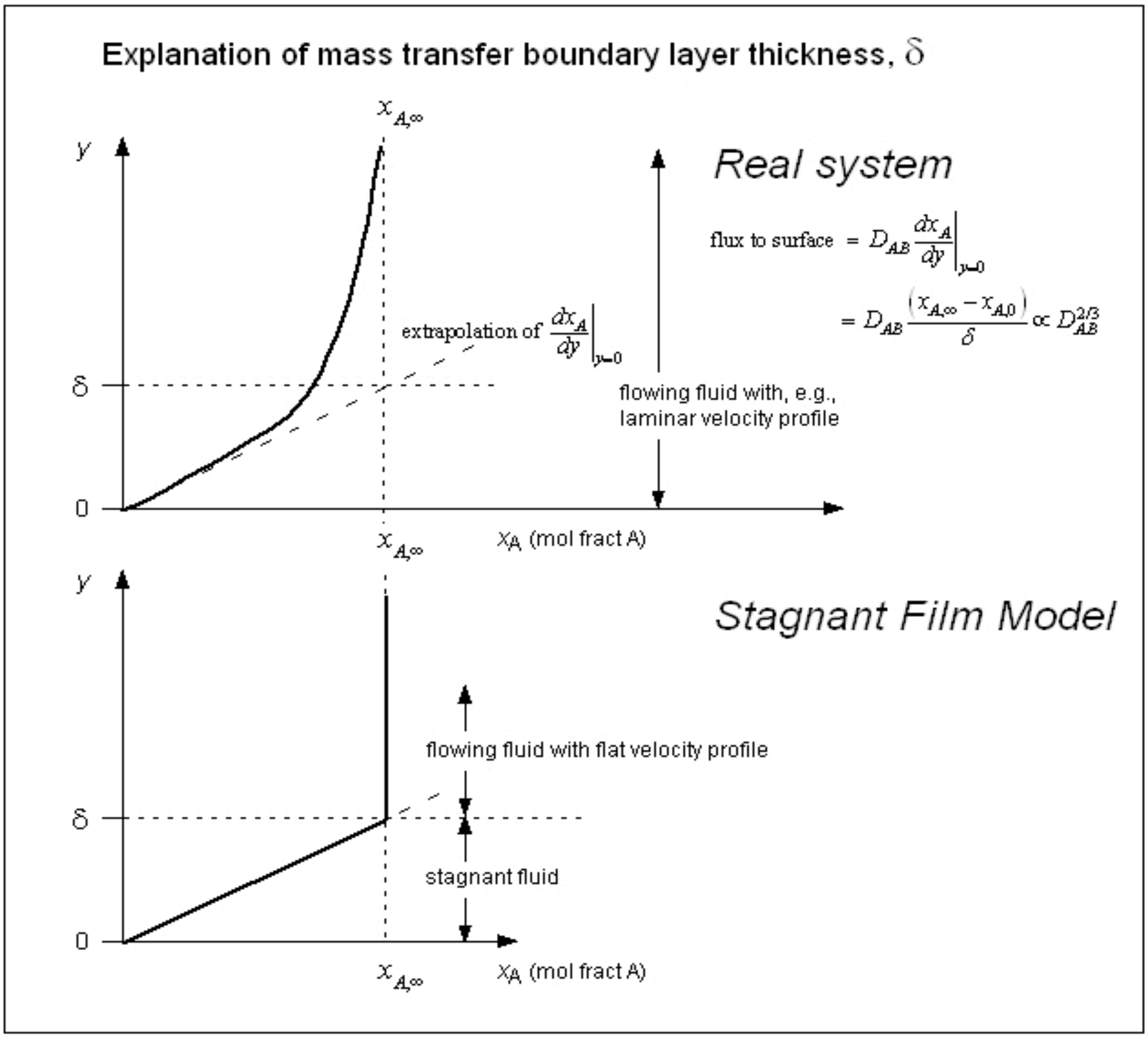
$$J_{A,wall} \propto \frac{D}{\delta_c} = \frac{D}{D^{1/3}} = D^{2/3}$$

For CVD with gas flowing over wafers laying on a susceptor, the decrease in mass transfer coefficient with distance from the leading edge of the susceptor will lead to non-uniformity in the growing film. This can be counteracted in at least two ways. First, by putting the wafers some distance away from the leading edge of the susceptor. Second, by sloping the susceptor in order to produce a downward component of velocity toward the surface ($u_y < 0$). This downward velocity component tends to decrease the changes in boundary layer thickness, mass transfer coefficient, and film growth rate. Burmeister (1993) shows the changes in similarity solution for heat transfer that starts some distance from the leading edge, and for flow over a wedge (Section 5.6).

Sometimes, when you hear people refer to a heat or mass transfer "film thickness," or heat or mass transfer "boundary layer thickness," they may talk as if there is a stagnant fluid boundary layer instead of the actual flowing boundary layer. The figure below explains that model.

We can see from the computed velocity profiles above, that a better description of the velocity boundary layer is that there is a roughly linear change in velocity over the velocity boundary layer thickness.

NOTE: In the figure below, mole fraction is x_A



Comparison to notation of Welty, et al. (2008) for the velocity profile.

Welty, et al. (2008)	**These notes**
$\eta = \frac{y}{2}\sqrt{\frac{U}{\nu x}}$	$\eta = \frac{y}{2}\sqrt{\frac{U}{\nu x}}$
υ_x	u_x
υ_y	u_y
$\sqrt{\nu x \upsilon_\infty}$	H
f also φ where they discuss shooting method	F
$f' = 2\frac{\upsilon_x}{\upsilon_\infty}$	$F' = 2\frac{u_x}{U}$
f''	F''
f'''	F'''
$f''' + f f'' = 0$	$F''' + F F'' = 0$
$f'(\eta \to \infty) = 2$	$F'(\eta \to \infty) = 2$
$\upsilon_x / \upsilon_\infty = 0.99 \; at \; \eta = 2.5$	$u_x / U = 0.99 \; at \; \eta = 2.5$

Comparison to notation of Middleman & Hochberg (1993) for the velocity profile.

Middleman & Hochberg (1993)	**These notes**
$\eta = \frac{y}{2}\sqrt{\frac{U}{\nu x}}$	$\eta = \frac{y}{2}\sqrt{\frac{U}{\nu x}}$
Λ	1
f	F
where $f = \int 2\left(\frac{u_x}{U}\right) d\eta = \int 2\,\Pi\, d\eta$	where $F = \int 2\left(\frac{u_x}{U}\right) d\eta = \int F'\, d\eta$
$\Pi = \frac{u_x}{U}$	$\frac{1}{2}F' = \frac{u_x}{U}$
Π'	$\frac{1}{2}F''$
Π''	$\frac{1}{2}F'''$
$\Pi'' + \Lambda f \Pi' = 0$	$F''' + F F'' = 0$

Comparison to notation of Burmeister (1993) for the velocity profile.

Burmeister (1993) ($\cdot_B$)	**These notes ($\cdot_H$)**
$\eta_B = y\sqrt{\frac{U}{\nu x}}$	$\eta_H = \frac{y}{2}\sqrt{\frac{U}{\nu x}}$
u	u_x
v	u_y
H	H
F_B	$2\,F_H$
$F_B{}' = \frac{u_x}{U}$	$F_H{}' = 2\frac{u_x}{U}$
$F_B{}''$	$2\,F_H{}''$
$F_B{}'''$	$2\,F_H{}'''$
$F_B{}''' + \frac{1}{2}F_B F_B{}'' = 0$	$F_H{}''' + F_H F_H{}'' = 0$
$F_B{}'(\eta_B \to \infty) = 1$	$F_H{}'(\eta_H \to \infty) = 2$
$u/U = 0.99\ at\ \eta_B = 5$	$u_x/U = 0.99\ at\ \eta_H = 2.5$

References

Burmeister, L. C., Convective Heat Transfer, 2nd ed., John Wiley & Sons (1993).

Middleman, S. and Hochberg, A, Process Engineering Analysis in Semiconductor Device Fabrication, McGraw-Hill (1993).

Welty, J. R., Wicks, C. E., Wilson, R. E., and Rorrer, G. L., Fundamentals of Momentum, Heat, and Mass Transfer, 5th Ed., John Wiley & Sons (2008).