Control of an inherently linear system - Part 2 - Proportional plus Integral action

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Consider the tank in SimzLab, Control Lab, Division 1, Lab 6, Temperature Control
www.SimzLab.com

Start with the open-loop transfer function equation that we derived earlier:

$$T(s) = G_P(s)T_j(s) + G_L(s)T_{in}(s)$$

$$G_p(s) = \frac{K_P}{(\tau_P s + 1)}$$

$$G_L(s) = \frac{K_L}{(\tau_P s + 1)}$$

Now add the feedback controller to the transfer function equation.

$$T(s) = G_1(s)R(s) + G_2(s)T_{in}(s)$$

where *R(s)* is the transform of the set point for the tank temperature, and where

$$G_1(s) = \frac{G_{sp}G_cG_aG_p}{1 + G_cG_aG_pG_m} = \frac{G_cG_p}{1 + G_cG_p} \quad \text{= transfer function of servo problem}$$

$$G_2(s) = \frac{G_L}{1 + G_cG_aG_pG_m} = \frac{G_L}{1 + G_cG_p} \quad \text{= transfer function of load problem}$$

$G_{sp} = G_a = G_m = 1$ are specified for this system,

For other systems, the actuator and measurement devices may have dynamic responses of their own such that their G's may be functions of the transform variable *s* and not constant gains as shown here.

Now we will use Proportional-Integral (PI) control. For PI control, the controller's output command signal in the time domain is:

$$p(t) \ = \ \text{manual bias} \ + \ K_c\left(e(t) + \frac{1}{\tau_I}\int_0^t e(t)\right)$$

$$p^{\Delta}(t) \ = \ K_c\left(e(t) + \frac{1}{\tau_I}\int_0^t e(t)\right)$$

For a hypothetical constant error value *e(t)* = 1, after time τ_I the integral-mode contribution to the command increases, or "resets," to equal to the proportional-mode contribution. Time τ_I is called the integral time or reset time. It's inverse is called the reset rate.

The controller transfer function is:

$$G_c = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

Putting G_c into the closed-loop transfer functions and doing some algebra, we get:

$$G_1(s) \;=\; \frac{(\tau_I s + 1)}{\left(\tau_{PI}^2 s^2 + 2\tau_{PI}\zeta s + 1\right)}$$ = transfer function of servo problem

Using the final value theorem and substituting $s = 0$ into $G_1(s)$, we see that, for a unit step change in set point, the steady-state gain is 1. That is good. We want a steady state gain of 1. That means that the tank temperature exactly follows the set point at long time.

$$G_2(s) \;=\; \frac{\left(\frac{\tau_I K_L}{K_c K_P}\right) s}{\left(\tau_{PI}^2 s^2 + 2\tau_{PI}\zeta s + 1\right)}$$ = transfer function of load problem

For a unit step change in load (fluid inlet temperature), the steady-state gain is 0. That is good. We want a steady state gain of 0. That means that the tank temperature returns to the set point after the disturbance.

We see from the denominator of the transfer functions that adding PI control to a first-order process results in a second-order system.

The terms above are as follows:

$$\tau_f = \left(\frac{V}{q}\right)$$ = time constant for fluid flow,

$$\tau_x = \left(\frac{\rho C_p V}{UA}\right)$$ = time constant for heat exchange,

$$\tau_P = \left[\frac{1}{\frac{1}{\tau_f} + \frac{1}{\tau_x}}\right]$$ = time constant for the overall open-loop process,

$$K_P \;=\; \left(\frac{\tau_P}{\tau_x}\right)$$ = open-loop gain for change in manipulated variable T_j, here $K_p = 0.5$

$$K_L = \left(\frac{\tau_P}{\tau_f}\right)$$ = open-loop gain for change in load or disturbance variable T_{in}, here $K_L = 0.5$

$$\tau_{PI}^2 = \left(\frac{\tau_I \tau_P}{K_c K_P}\right) = \left(\frac{\tau_I \tau_x}{K_c}\right)$$ = square of time constant of closed-loop system with PI control,

$$2\tau_{PI}\zeta = \tau_I\left(1+\frac{1}{K_c K_P}\right)$$

$$\zeta = \frac{1}{2}(1+K_c K_P)\sqrt{\frac{\tau_I}{K_c K_P \tau_P}}$$ = damping ratio of closed-loop system with PI control,

$$\zeta = \frac{1}{2}\left(1+\frac{\tau_x}{K_c \tau_P}\right)\sqrt{\frac{K_c \tau_I}{\tau_x}}$$

By varying the controller values of proportional gain, K_c , and integral time τ_I , we can get

- over damped response, two distinct real roots, $\zeta > 1$
- critically damped response, two repeating real roots, $\zeta = 1$
- under damped response, two complex-conjugate roots, $0 < \zeta < 1$
- undamped response, two purely imaginary conjugate roots, $\zeta = 0$

You can also see how we can get transfer functions with polynomials in both the numerator and the denominator, as shown in the discussion of partial-fraction expansion of transfer functions.

Now you can go the the SimzLab simulator and explore these conditions and see the responses.

See Section 3.2 of Chau's textbook. Also see Example 5.3, p. 99, of Chau's textbook.

Putting G_c into the closed-loop transfer functions and doing some algebra, we get:

$$G_1(s) = \frac{K_1 K_c(\tau_I s+1)\left(\left(\frac{\tau_{PI}^2}{\tau_I}\right)s+\left(\frac{1}{K_c K_1}\right)\right)}{(\tau_P s+1)\left(\tau_{PI}^2 s^2+2\tau_{PI}\zeta s+1\right)}$$

$$G_1(s) = \frac{(\tau_I s+1)(\tau_P s+1)}{(\tau_P s+1)\left(\tau_{PI}^2 s^2+2\tau_{PI}\zeta s+1\right)}$$ transfer function of servo problem

$$G_1(s) = \frac{(\tau_I s+1)}{\left(\tau_{PI}^2 s^2+2\tau_{PI}\zeta s+1\right)}$$ transfer function of servo problem

$$G_2(s) = \frac{\left(\frac{\tau_P}{\tau_f}\right)\left(\tau_{PI}^2 s^2 + \left(\frac{\tau_I}{K_c K_1}\right) s\right)}{(\tau_P s + 1)\left(\tau_{PI}^2 s^2 + 2\tau_{PI}\zeta s + 1\right)}$$ transfer function of load problem

$$G_2(s) = \frac{\left(\frac{\tau_I \tau_P}{K_c K_1 \tau_f}\right) s}{\left(\tau_{PI}^2 s^2 + 2\tau_{PI}\zeta s + 1\right)}$$ transfer function of load problem

$$G_2(s) = \frac{\left(\frac{\tau_I \tau_x}{K_c \tau_f}\right) s}{\left(\tau_{PI}^2 s^2 + 2\tau_{PI}\zeta s + 1\right)}$$ transfer function of load problem