In this course we consider linear process models. The Laplace transform can be used with linear differential equations. So what happens when our process model is nonlinear? We have to "linearize" the model about an initial steady state. We will do this using the first-order Taylor approximation (Taylor series approximation, Taylor expansion).

Below, f(x) is a function of the variable x, and x varies as a function of time, t. The initial condition at t = 0 is a steady state that we specify. This initial steady state is denoted by the subscript s. The function f(x,y) is a function of two variables. Even if the orginal function is nonlinear in x and/or y, the approximation on the right-hand sides below are linear in x and y.

$$f(x) = f(x_s) + \left.\frac{df}{dx}\right|_s (x - x_s)$$

WHERE

$$x = x(t)$$
$$x_s = x(0)$$

$$f(x,y) = f(x_s, y_s) + \left.\frac{\partial f}{\partial x}\right|_s (x - x_s) + \left.\frac{\partial f}{\partial y}\right|_s (y - y_s)$$

WHERE

$$x = x(t), \quad y = y(t)$$
$$x_s = x(0), \quad y_s = y(0)$$

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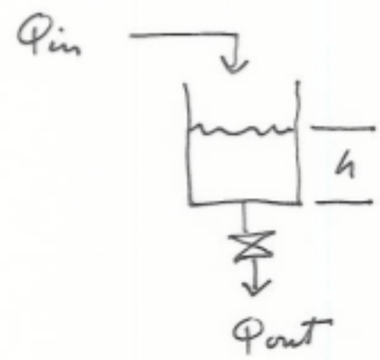
Consider liquid flowing into and out of a tank. The inlet flow rate can vary with time. The outlet flow rate is proportional to the square root of the hydrostatic head. Thus, the outlet flow rate may be different than the inlet flow rate, causing the height of liquid in the tank to vary with time.



TOTAL MASS BAL

$$\rho \frac{dV}{dt} = \rho q_{in} - \rho q_{out}$$

where $q_{out} = c_v \left(\rho \left(\frac{g}{g_c} \right) h \right)^{1/2}$

$$V = A h$$

$$A \frac{dh}{dt} = \varphi_{in}(t) - \underbrace{c_v \left(\rho \frac{g}{g_c}\right)^{1/2}}_{\beta} h^{1/2}$$

The equation is nonlinear in the liquid height, h. Now apply the first-order Taylor approximation to the $h^{1/2}$ term about an initial steady-state value of the height, h_s.

$$f(h) = h^{1/2} \approx h_s^{1/2} + \left.\frac{\partial f}{\partial h}\right|_s (h - h_s)$$

$$\approx h_s^{1/2} + \left(\frac{1}{2} h_s^{-1/2}\right)(h - h_s)$$

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Remember that the initial steady-state value of h, h_s, is a constant. We now have a linearized equation:

$$A\frac{dh}{dt} = Q_{in}(t) - \beta\left[h_s^{1/2} + \left(\frac{1}{2}h_s^{-1/2}\right)(h-h_s)\right]$$

Define deviation variables. Here they are shown with a superscript delta:

DEFINE DEVIATION VARIABLES ($^\Delta$)

$$h^\Delta = h - h_s \Rightarrow h = h^\Delta + h_s \Rightarrow \frac{dh}{dt} = \frac{dh^\Delta}{dt} + \cancel{\frac{dh_s}{dt}}^{\,0}$$

$$Q_{in}^\Delta = Q_{in} - Q_{in,s} \Rightarrow Q_{in} = Q_{in}^\Delta + Q_{in,s}$$

AT STEADY STATE ($_s$)

$$A\frac{dh_s}{dt} = 0 = \varphi_{IN,s} - \beta h_s^{1/2}$$

SUBTRACT SS EQN FROM ABOVE

$$\frac{dh^\Delta}{dt} + \underbrace{\left(\frac{\beta}{2A h^{1/2}}\right)}_{q} h^\Delta = \frac{\varphi^\Delta_{IN}(t)}{A}$$

The equation is linear in the deviation variables. Now apply the Laplace transform:

$$sH(s) + q H(s) = \left(\frac{1}{A}\right) \varphi_{IN}(s)$$

$$H(s) = \left(\frac{1}{s+q}\right)\left(\frac{1}{A}\right) \varphi_{IN}(s)$$

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Let's compare the predicted response of the linearized model to an experiment in SimzLab's Control Lab, Division 1, Lab 3.

Run the lab under manual control, establish an initial level of 0.99 m at an inlet flow rate of 1.40 m^3/min, pause the simulation, and then make a step change in inlet flow rate to 2.40 m^3/min (a deviation in inlet flow rate of +1.0 m^3/min), and then continue to run the simulation.

$$\varphi_{IN}^{\Delta}(t) = 1 \ m^3/min$$

$$\Phi_{IN}(s) = \frac{1}{s}$$

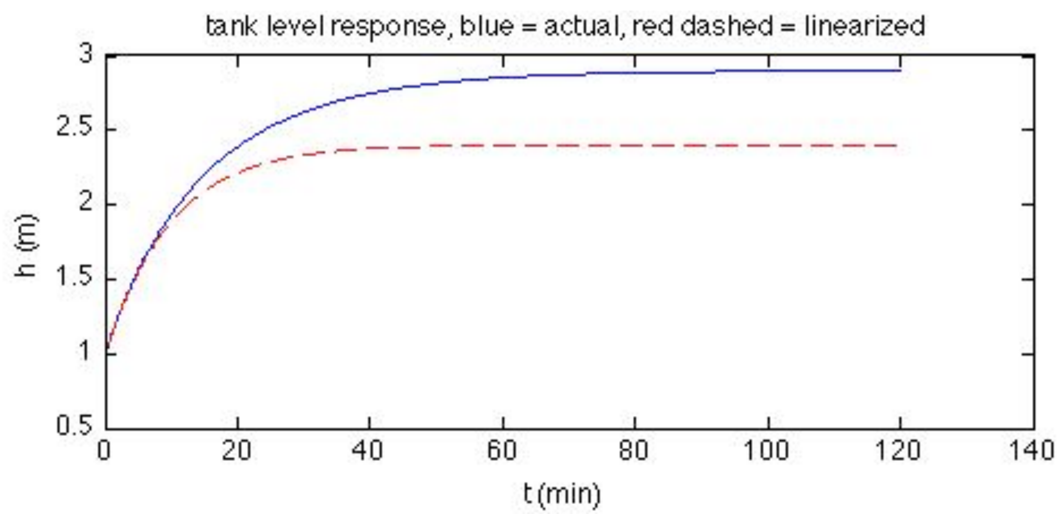
$$H(s) = \frac{(1/A)}{s(s+\varphi)}$$

We can lookup the inverse transform in the table.

$$h^{\Delta}(t) = \left(\frac{1}{A}\right)\left(\frac{1}{\varphi}\right)\left(1 - e^{-\varphi t}\right)$$

Parameter values in SimzLab were: A = 7.07 m^2, beta = 1.41 $m^{5/2}$/min, alpha = 0.100 (1/min). Accounting for the units in Q_{in}, the units of h are meters.

The plot below compares the actual, nonlinear response to the response of the linearized model.



The plot reminds us that the linearized model is only an approximation.

You can see that there is reasonable agreement at early time, when the deviation from the initial state is less than 0.7 m.

The next few cards have a listing of the Matlab program that generated the plot shown on the previous card:

```
% SimzLab, Control Lab D1L3
% Compare actual nonlinear response to
% response of linearized approximation

clear all
% draw a line in command window to separate
% runs when using > button in editor to run
fprintf('------------------- \n')

Cv = 10; % m3/min/bar^(1/2), maximum valve flow coefficient
command = 0.55; % manual controller command signal for this experiment
Cv = Cv * ( 1 - command);  % Cv for linear, reverse-acting valve

D = 3; % m, tank diameter
A = pi*(D/2)^2; % m2, cross-sectional area of tank

g = 9.807; % m/s2, gravitational acceleration
gc = 1.0; % (s2 N)/(kg m), SI force unit conversion factor
rho = 1000; % kg/m3, density of water

qin = 2.40; % m3/min, final inlet flow rate
```

```
i = 1; % array index
t(i) = 0; % s, initial time

h(i) = 0.99; % m, initial height
dp(i) = rho*g/gc/1e5*h(i); % bar, initial hydrostatic head
qout(i) = Cv * sqrt(dp(i)); % m3/min, initial outlet flow rate
dt = 0.1; % min, time step
tf = 120; % min, final time

% integrate nonlinear model with Euler's method, OK for our purposes here
while t(i) < tf
    dhdt = 1/A * (qin - qout(i));
    h(i+1) = h(i) + dhdt*dt;
    t(i+1) = t(i) + dt;
    dp(i+1) = rho*g/gc/1e5*h(i+1);
    qout(i+1) = Cv * sqrt(dp(i+1));
    i = i+1;
end

% now compute predicted response of linearized model

beta = rho*g/gc; % N/m2/m = Pa/m
beta = beta/1e5; % bar/m, where 1e5 Pa/bar
beta = Cv * sqrt(beta); % m3/min/m^(1/2)
```

```
qinInitial = 1.40; % m3/min, initial inlet flow rate
qinFinal = 2.40; % m3/min, final inlet flow rate
qinDev = qinFinal - qinInitial; % m3/min, deviation in inlet flow rate

% first get analytical solution of linearized model
% from Laplace transform

alpha = beta/2/sqrt(h(1))/A;
hDev = qinDev/A/alpha * (1 - exp(-alpha*t));
hLin = h(1) + hDev;

% check by integrating linearized model with Euler's method

i = 1; % array index
t(i) = 0; % s, initial time
hDev2(i) = 0; % m, initial height deviation
dt = 0.1; % min, time step
tf = 120; % min, final time
while t(i) < tf
    dhDev2dt = qinDev/A - alpha*hDev(i);
    hDev2(i+1) = hDev2(i) + dhDev2dt*dt;
    t(i+1) = t(i) + dt;
    i = i+1;
end
```

```
hLin2 = h(1) + hDev2;

plot(t,h,'b',t,hLin,'r--') % ,t,hLin2,'g')
title('tank level response, blue = actual, red dashed = linearized')
ylabel('h (m)')
xlabel('t (min)')
```