To this point, we have considered simple processes: concentration and level variations in tanks.

The differential equations that described those processes, after linearization when necessary, were first-order differential equations. Such processes are termed "first-order processes."

The transfer functions had a numerator that was a constant and a denominator that was a first-order polynomial in the transform variable s.

This polynomial in the denominator of the transfer function is the characteristic equation or characteristic polynomial of the process.

That is, it is the polynomial in s that results from applying the Laplace transform to the output part (left-hand side) of the differential equation that describes the process. The "forcing function" is on the right-hand side.

For deviation variables, s is a "marker" of a first-derivative of the output variable with respect to time, and s^2 is a marker of a second-derivative.

When we get to more complex processes, especially those that involve feedback control, we will get more complex transfer functions.

rherz@ucsd.edu

Complex dynamic responses

Even for complex systems, we can represent our general problem in the frequency domain as:

$$Y(s) = G(s)X(s)$$

where

$$Y(s) = \text{transform of output}$$

$$X(s) = \text{transform of input}$$

$$G(s) = \text{transfer function} = \text{transform of unit impulse response}$$

Knowing the transfer function G(s), and given an input transform X(s), we get the output transform Y(s).

The problem is to transform Y(s) back to the output response y(t) in the time domain.

First, let's focus on the transfer function G(s), which is the transform of the response of the system to a unit impulse input.

Since any arbitrary input x(t) can be represented as a series of impulse inputs of varying amplitude and time delay, knowing the unit impulse response g(t), and therefore its transform G(s), is sufficient information to obtain the response to any arbitrary input.

on p. 22 of Chau's textbook, we see

$$Y(s) = G(s)X(s)$$

$$\frac{Y(s)}{X(s)} = G(s)$$

A transfer function G(s) can be represented by a ratio of two polynomials

$$G(s) = \frac{Q(s)}{P(s)}$$

on p. 16 of Chau's textbook, we see

$$F(s) = \frac{q(s)}{p(s)}$$

This is just a change in notation. These mean the same thing

$$G(s) = \frac{Q(s)}{P(s)} = F(s) = \frac{q(s)}{p(s)}$$

You are going to encounter changes in notation within this textbook and between different textbooks and papers. Therefore, you are going to have to understand the material well such that you can adapt to changes in notation.

The simple processes we have considered so far have a constant in the numerator of the transfer function, rather than a polynomial.

When we add feedback control, we will get a polynomial in the numerator.

Chau's textbook focuses on math first and then on physical processes and applications.

Therefore, we are going to learn about how to handle complex transfer functions before we see a system in which they will appear.

The polynomial order of q(s) is m. The polynomial order of p(s) is n. For real, physical systems, m < n.

$$G(s)=\frac{Q(s)}{P(s)}=F(s)=\frac{q(s)}{p(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots a_1 s+a_0}$$

A transfer function can be factored into a sum of simple terms. This is related to the fact that a high-order ordinary differential equation (ODE) can be factored into a system of coupled first-order ODE's.

Case 1. For the case of distinct real roots

$$G(s)=\frac{\alpha_1}{s+a_1}+\frac{\alpha_2}{s+a_2}+\ldots+\frac{\alpha_n}{s+a_n}$$

$$\alpha_j \;=\; \text{constant to be determined}$$

$$a_j \;=\; -p_j \;=\; \text{negative of root j of polynomial p(s)}$$

For stable systems (vs. undesirable unstable systems), $a_j > 0$ (negative root $p_j < 0$) such that the transform of each of the terms in the sum are exponential decays: $\alpha_j e^{-a_j t}$

Case 2. Complex conjugate roots. These may appear alone, as shown below, or in combination with real roots.

$$G(s) = \frac{\alpha}{s+a} + \frac{\alpha*}{s+a*} = \frac{\alpha}{s-p} + \frac{\alpha*}{s-p*}$$

The complex conjugate roots are

$$p = -u+\omega j \quad \text{or} \quad -u+\omega i$$

$$p* = -u-\omega j \quad \text{or} \quad -u-\omega i$$

where the square root of -1 can be represented by j or i. The inverse transform, after applying Euler's identity and the trigonometric identity between a sum of sines and cosines and the sine-phase-lag form, is:

$$g(t) = \beta e^{-ut}(\sin(\omega t)+\phi)$$

$$\phi = \text{phase lag or phase shift}$$

That is, complex roots result in an oscillatory response. In a desirable stable system, the real part of the root is negative ($-u < 0$) such that the oscillatory response to an impulse input decays with time.

Case 2 continued...complex roots..

On the previous card, in order to convert from the complex roots to the sine-phase-lag form, use,

Euler's identity,

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

and this trigonometric identity,

$$a \sin(\omega t) + b \cos(\omega t) = \left(\sqrt{a^2+b^2}\right)\sin(\omega t+\phi)$$

where the phase lag or phase shift is

$$\phi = \tan^{-1}(b/a) + 0 \text{ for } a \geq 0 \text{ OR } +\pi \text{ for } a<0$$

See the example on p. 19 of Chau's textbook. The inverse tangent or arctangent function in Matlab is atan( ).

Case 2 continued...complex roots...

On the first card for this case, we saw G(s) separated into two first order terms with complex roots. An alternate and more common way to represent this situation is to keep G(s) as a second-order term:

$$G(s) = \frac{K}{\tau^2 s^2 + 2\tau\zeta s + 1} = \frac{\alpha_1}{s - p_1} + \frac{\alpha_2}{s - p_2} = \frac{\alpha_1}{s + a_1} + \frac{\alpha_2}{s + a_2}$$

$$p_{1,2} = -\frac{\zeta}{\tau} \pm \frac{\sqrt{\zeta^2 - 1}}{\tau}$$

For $\zeta > 1$ (zeta > 1) there are two distinct real roots resulting in stable, exponential decays. This case is called "over-damped." This second-order transfer function is equivalent to Case 1 above.

For $\zeta = 1$ there are two repeating roots resulting in stable, extended decays. This case is called "critically damped." This second-order transfer function is equivalent to Case 3 below.

For $0 < \zeta < 1$ there are two complex-conjugate roots resulting in stable, decaying oscillations with frequency $\omega = \left(\sqrt{1 - \zeta^2}\right) / \tau$. This case is called "under-damped."

For $\zeta = 0$ there are two purely imaginary conjugate roots resulting in sustained oscillations with "natural frequency" $\omega_n = 1/\tau$. This case is called "undamped." See p. 49 of Chau's textbook.

Case 3. For repeated roots, e.g., for the case of one root ($-a_1$) plus a second root ($-a_2$) that is repeated twice, for example,

$$G(s) = \frac{q(s)}{(s+a_1)(s+a_2)^2} = \frac{\alpha_1}{s+a_1} + \frac{\alpha_2}{s+a_2} + \frac{\alpha_3}{(s+a_2)^2}$$

The inverse transform of this example is

$$g(t) = \alpha_1 e^{-a_1 t} + (\alpha_2 + \alpha_3 t) e^{-a_2 t}$$

The effect of the repeated root is a slower decay with time than if the root were not repeated.

See these cases in the textbook for derivations and examples.

The important results:

In all cases, we can factor a complex transfer function into a sum of simple terms. These terms are first-order terms or second-order terms.

The inverse transform of each of the simple terms can be looked up in a table of Laplace transforms.

Finally, the unit impulse response of a complex system in the time domain will be a sum of terms of just a few different types.

The time responses - decay rates and oscillation frequencies - are determined by the roots of the characteristic polynomial in the denominator of the transfer function.

The polynomial in the numerator affects the amplitudes of the responses.

Complex dynamic responses

OK, now we know that the unit impulse response of any stable linear system will be the sum of one or more of the following types of terms:

exponential decay: $\alpha_j e^{-a_j t}$

oscillating decay: $\beta e^{-ut}(\sin(\omega t)+\phi)$

extended decay: $(\alpha_0+\alpha_1 t+\alpha_2 t^2+\ldots)e^{-a_j t}$

This is because even a complex transfer function can be converted by partial fraction expansion into a sum of only a few different types of simple terms.

Unstable systems, which can occur but which are undesirable, grow in magnitude exponentially because they have positive real roots of their characteristic equation, the denominator polynomial p(s).

If an impulse input is not of unit magnitude, the output will be the unit impulse response multiplied by the magnitude of the input.

Complex dynamic responses

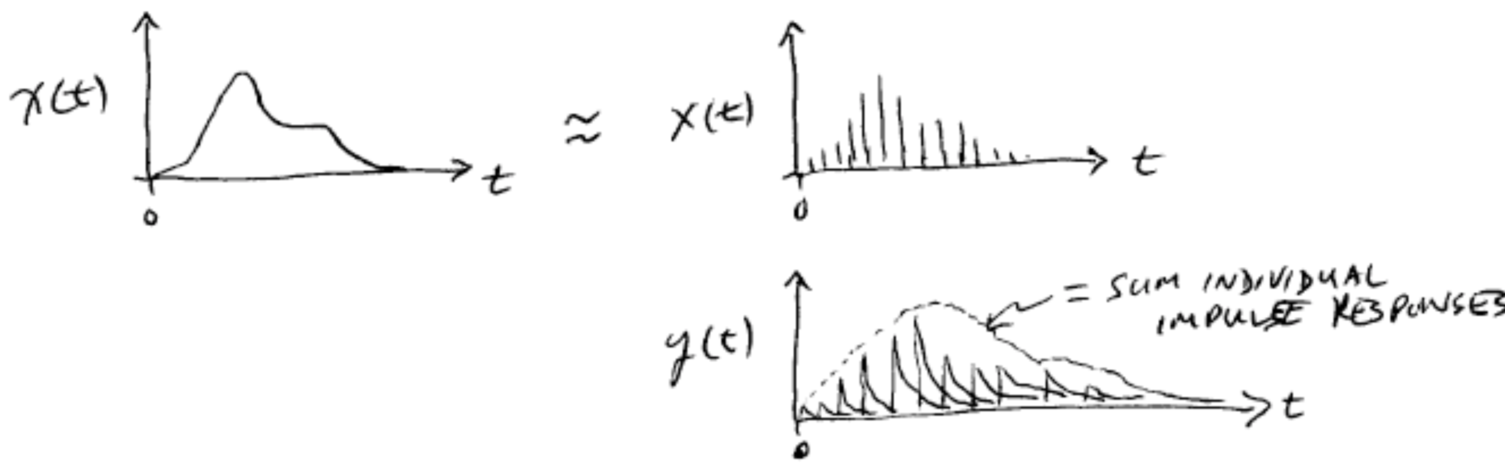
Now, what happens when the input to a system is something other than an impulse (Dirac delta)?

Any arbitrary input, or forcing function, can be represented as a series of impulse inputs of varying amplitude and time delay.

Therefore, knowing the unit impulse response of a system is sufficient information to obtain the response to any arbitrary input.

In theory, all we would have to do would be to sum the responses to each of the input impulses to obtain the complete output response.

We can sum individual responses to obtain the complete response because we are dealing with linear systems.



How can we obtain the unit impulse response of a system?

1) Write a math model of the system and do the math, and/or

2) do experiments and analyze the output response. For example, use as an input to the system

a) an approximate impulse, which is a finite input over a time period that is short with respect to the smallest characteristic time constant of the system, or

b) a step change input.

rherz@ucsd.edu

Complex dynamic responses

After having separated a transfer function into a sum of terms, we just need to (1) multiply each of the terms by the transform of the input, and (2) transform each product back to the time domain.

$$G(s)=G_1(s)+G_2(s)+G_3(s)+\ldots$$

$$Y(s)=G(s)X(s)$$

$$Y(s)=G_1(s)X(s)+G_2(s)X(s)+G_3(s)X(s)+\ldots$$

$$y(t)=y_1(t)+y_2(t)+y_3(t)+\ldots$$

In most cases when designing control systems, we are going to be dealing with only a few types of inputs: (1) impulse, (2) step change, (3) sinusoidal input (sine wave).

The exception is in homework and quiz problems where anything can be an input!

For a sinusoidal input, we will be interested in the long term response only when we study "frequency response analysis." For stable systems, the long term output will be an sinusoid of the same frequency as the input and possibly shifted in time (phase shift) and possibly of a different amplitude.

A transfer function can be written in several equivalent forms:

$$G(s)=\frac{Q(s)}{P(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots a_1 s+a_0}$$

$$G(s)=\frac{Q(s)}{P(s)}=G_1(s)+G_2(s)+G_3(s)+\ldots$$

$$G(s)=\frac{Q(s)}{P(s)}=\frac{\alpha_1}{s+a_1}+\frac{\alpha_2}{s+a_2}+\ldots+\frac{\alpha_n}{s+a_n}$$

for example, with distinct real roots

$$G(s)=\frac{Q(s)}{P(s)}=\left(\frac{b_m}{a_m}\right)\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

"pole - zero form"

$$G(s)=\frac{Q(s)}{P(s)}=\left(\frac{b_0}{a_0}\right)\frac{(\tau_a s+1)(\tau_b s+1)\ldots(\tau_m s+1)}{(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)}$$

"time constant form"

Consider the pole-zero form:

$$G(s)=\frac{Q(s)}{P(s)}=\left(\frac{b_m}{a_m}\right)\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

The roots of the characteristic equation, the polynomial P(s), are the p's. When the value of the transform variable s equals the value of one of the p's, then you have a zero in the denominator and the magnitude of G(s) is infinite. So on a plot of G(s), you would see a "pole" sticking up on the plot. Thus, the "pole" in "pole-zero form."

The roots of the polynomial Q(s) are the z's. When the value of s equals the value of one of the z's, then the magnitude of G(s) is zero. Thus, the "zero" in "pole-zero form."

The time responses - decay rates and oscillation frequencies - are determined by the poles, which are the roots of the characteristic polynomial in the denominator of the transfer function.

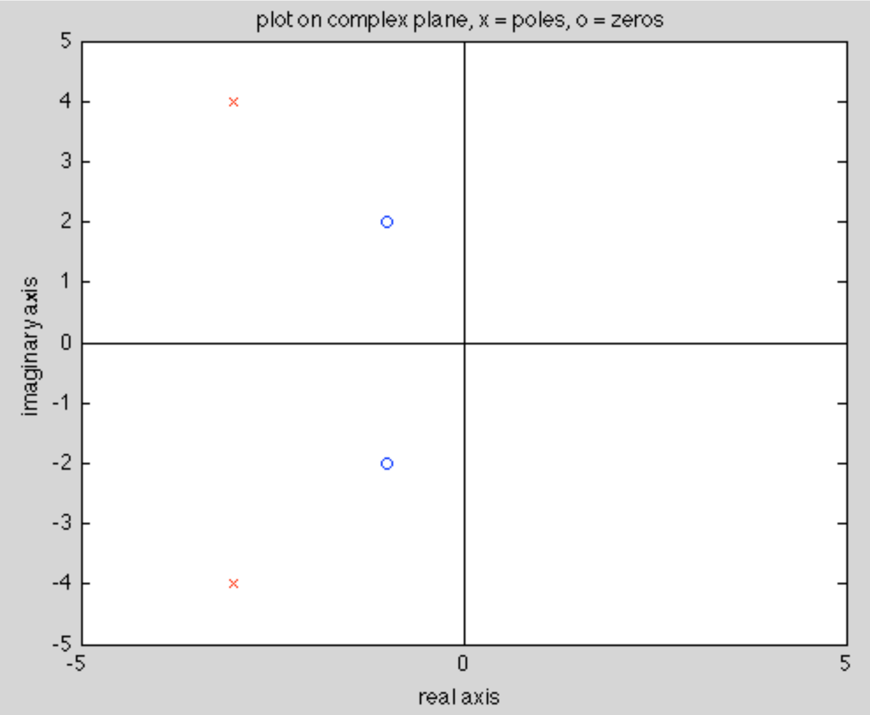
The zeros, which are the roots of the polynomial in the numerator, affect the amplitudes of the responses.

We are going to plot the location of poles and zeros when we do "root-locus analysis" in Chapter 7 of Chau's textbook. A root-locus plot is one method of characterizing and analyzing the dynamics of a system.

Complex dynamic responses

We plot poles and zeros on the complex plane. You can do this by hand, or by using Matlab. Stable systems, those whose poles have negative real parts, are desirable in control since disturbances decay. Poles of stable systems lie on the left-hand side of the complex plane plot.

```
z = [-1+2i, -1-2i]; % zeros
p = [-3+4i, -3-4i]; % poles (roots of charac eqn)
plot(z,'bo')
title('plot on complex plane, x = poles, o = zeros')
xax = [-5 5];
yax = xax;
axis([xax yax])
hold  % toggle hold on so can add more
plot(p,'rx')
plot(xax,[0 0],'k')
plot([0 0],yax,'k')
ylabel('imaginary axis')
xlabel('real axis')
hold  % toggle hold off
```



Consider the time constant form:

$$G(s)=\frac{Q(s)}{P(s)}=\left(\frac{b_0}{a_0}\right)\frac{(\tau_a s+1)(\tau_b s+1)\ldots(\tau_m s+1)}{(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)}$$

The dimension of the transform variable s is (1/time = frequency). The dimension of the time constants, the tau's, is (time).

The time constants of P(s) are related to the roots or poles of P(s): $\tau_j=-1/p_j$

For stable systems, the poles are negative and the time constants are positive.

The Final Value Theorem gives the final output of a stable system (Chau p. 15):

$$\lim\,[sF(s)]_{s\to 0}=\lim\,[f(t)]_{t\to\infty} \quad \text{or} \quad \lim\,[sY(s)]_{s\to 0}=\lim\,[y(t)]_{t\to\infty}$$

For a unit step input, X(s) = 1/s, we can see from the time constant form, that the steady state output value, the "steady-state gain," will be:

$$\lim\,[G(s)]_{s\to 0}=\lim\,[y(t)]_{t\to\infty}=\left(\frac{b_0}{a_0}\right) \quad \text{i.e., the 'steady-state gain'}$$