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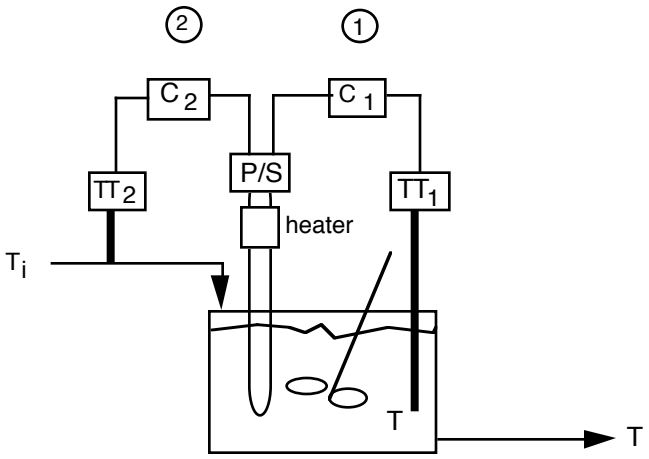
## 1. Introduction to Process Control and Process Models

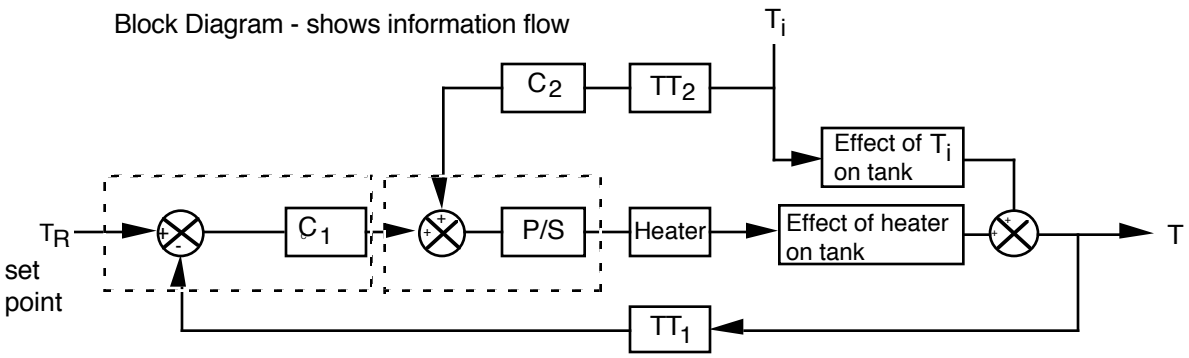
① Feedback Control - measure the control (output) variable and take action to bring control variable back to set point. Can't get "perfect" control since action taken only when there are devations from the set point, but handles changes in any input variable.

② Feedforward Control - measure an input variable and take action to keep control variable at set point. Can get "perfect" control but only corrects for change in the measured input variable, not changes in others.

Schematic Diagram




Block Diagram - shows information flow



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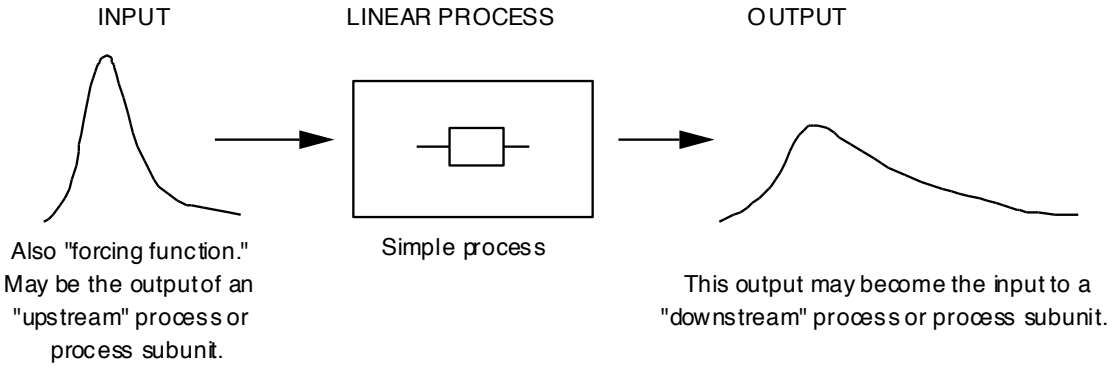
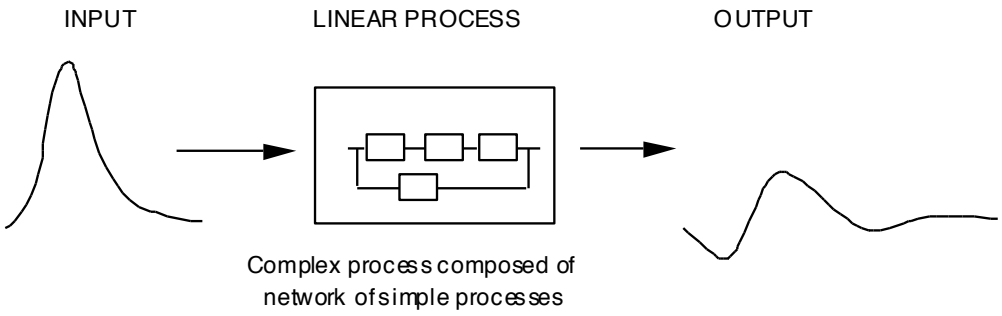
## Our Problem Statement:

We consider linear processes (processes that can be described by linear differential equations with constant coefficients) or nonlinear processes that have been linearized about the normal steady-state operating conditions.

There may be several inputs (independent variables) and several outputs (dependent variables). Here, we consider a change in one input variable and the response of one output variable. Processes may be simple or may be complex networks of simple processes:




There are three elements to the problem: input, process, output. In general, we will know two elements and have to find the third. Specific objectives will be to:

(1) predict the output, given an input and a process,

(2) design a control system for a process that will give the desired output in response to a given input (e.g., maintain current output, within a specified band, in response to a unit step input),

(3) determine what the internal structure of the process is, given pairs of inputs and outputs.

Suggested procedure for developing process models:

1. Draw process schematic ("sketch"), plus block diagram for control problems.

2. Label sketch - variables and parameters with dimensions ("units").

3. List and justify assumptions.

4. Identify independent (input) and dependent (control, output) variables.

5. Identify "control volume" for conservation equations ("balances").

6. Write conservation equations:

   1) word equation first (e.g., accum = in - out + generation)
   2) write math terms under word equation
   3) check units (helps to catch any parameters left out)
   4) check signs (set all but 2 terms to zero and see if signs make sense, repeat until all terms are checked)

7. Check degrees of freedom, $N_F$, to see if system is exactly specified:

   $N_F = N_V - N_E = 0$? where $N_V$ = no. of variables and $N_E$ = no. of equations,
   where $N_E$ includes conservation equations and equations specifying the time variation of input variables.

8. Linearize nonlinear equations if necessary.

9. Define dimensionless variables, or dimensionless deviation variables for control problems, and make equations dimensionless. Write a word definition of the dimensionless parameter groups which result.

Reasons for making equations dimensionless:

- simplifies writing equations, reduces clutter
- easier to see general mathematical form of equation
- get insight from dimensionless groups, see that only the parameter groups are important, not individual parameters
- reduces the number of times you have to solve equations

Summary of Equations for Simple Heated Stirred Tank:

Assume that the response of electrical heating elements is "fast" with respect to the response of the liquid temperature in the tank (i.e., elements have negligible heat capacity).

Input Variables: $T_i(t)$ [K, temperature of inlet fluid], Q(t) [J/s, heat input from electrical heater element], $T_a(t)$ [K, average temperature of heat transfer fluid over area A]

Output Variable: T(t) [K, temperature of fluid in tank and outlet fluid].

Constants: w [kg/s, mass flow rate], V [$m^3$, volume of fluid], ρ [$kg/m^3$, density of fluid], C [J/(kg · K), average mass heat capacity of fluid], U [J/(s · $m^2$ · K), heat transfer coefficient over area A], A [$m^2$, heat transfer area].

Energy Balance on Fluid in Tank:

$$\rho VC\frac{dT}{dt} = wC\left(T_i - T\right) + Q - UA\left(T - T_a\right) \tag{1-1}$$

Energy balance with all variables except time t made into dimensionless deviation variables:

$$\frac{dT^{\Delta^*}}{dt} + \left(\phi + \gamma\right)T^{\Delta^*} = \phi T_i^{\Delta^*} + \phi Q^{\Delta^*} + \gamma T_a^{\Delta^*} \tag{1-2}$$

initial conditions: at $t = 0, T^{\Delta^*} = 0, T_i^{\Delta^*} = 0, Q^{\Delta^*} = 0, T_a^{\Delta^*} = 0$

Dimensionless Deviation Variables:

superscript $^{\Delta}$ represents a deviation variable, deviation from normal steady-state

bar over variable represents value at normal steady-state conditions

superscript $^{*}$ represents a dimensionless variable

note there are ways to define these variables other than those shown here

$$T^{\Delta^*} = \frac{T - \overline{T}}{T_i^{\max} - \overline{T}} \qquad T_i^{\Delta^*} = \frac{T_i - \overline{T_i}}{T_i^{\max} - \overline{T}} \qquad T_a^{\Delta^*} = \frac{T_a - \overline{T_a}}{T_i^{\max} - \overline{T}} \qquad Q^{\Delta^*} = \frac{Q - \overline{Q}}{wC\left(T_i^{\max} - \overline{T}\right)}$$

$\phi = \dfrac{w}{\rho V}$ = rate coefficient for heating/cooling by flow in and out (1/s)

$\gamma = \dfrac{UA}{\rho VC}$ = rate coefficient for heating/cooling by heat transfer (1/s)

Time t can also be made dimensionless, as shown below, but we usually don't do that in process control.

## 2. Solution of Linear Differential Equations

### 2.1. Linear Differential Equations

In this course we will learn some methods for analyzing the dynamics of physical systems. By "dynamics," we mean how the systems change with time. Specifically, we will learn tools for analyzing linear dynamic systems. Linear dynamic systems vary with time and can be described by linear differential equations. A differential equation of order n is linear if it can be written in the following form:

$$y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + a_{n-2}(t)\, y^{(n-2)}(t) + \ldots + a_2(t)\, y''(t) + a_1(t)\, y'(t) + a_0(t)\, y(t) = f(t) \qquad (2.1\text{-}1)$$

where y', y'', $y^{(n-1)}$, and $y^{(n)}$ are derivatives of y(t), with respect to t, of order 1, 2, n-1, and n, respectively, and the $a_n(t)$ are coefficients that are functions of t only.

**The equation is linear because the dependent variable y(t) and its derivatives do not appear in products with themselves or with other variables.**

For example, the terms $y^2$, yy', or x(t)y(t) do not appear. The independent variable t will refer to time in this course, although this is not necessarily the case, in general. The term on the right hand side, f(t), does not contain the dependent variable y(t) or any of its derivatives and is called the nonhomogeneous term or "forcing function."

The primary tool we will become familiar with for solving linear differential equations, the Laplace transform, is usually only suitable for solving linear differential equations <u>with constant coefficients</u>:

$$y^{(n)}(t) + a_{n-1}\, y^{(n-1)}(t) + a_{n-2}\, y^{(n-2)}(t) + \ldots + a_2\, y''(t) + a_1\, y'(t) + a_0\, y(t) = f(t) \qquad (2.1\text{-}2)$$

**"Real systems are nonlinear, to varying degrees."**

We will consider perturbations of systems from initial operating conditions and control of systems about desired operating conditions. For "small" deviations from the initial or desired operating conditions, real systems can be considered approximately linear. The more nonlinear a system is, the smaller the deviations must be for approximately linear behavior.

For relatively linear systems, the deviations for which the system is approximately linear can be relatively large, and reasonable assumptions can be made such that linear conservation equations can be set up at the start of the analysis. However, it is always advisable to remember that nonlinear behavior will be obtained when large perturbations are applied.

For relatively nonlinear systems, the conservation equations set up at the beginning of the analysis will be nonlinear. In these cases, the original equations must be linearized about the initial or desired operating conditions in order to apply the methods discussed below. The

linearized equations will provide a good approximate description of the dynamic behavior of the system only for small deviations from the initial or desired operating conditions.

## 2.2. Linearization

Consider a system described by the following equation in which y(t) is the output variable (e.g., temperature of fluid in a tank), x(t) is the input variable (e.g., heater input), and f(y,x) is a nonlinear function of y and x:

$$\frac{dy}{dt} = f(y,x) \tag{2.2-1}$$

This equation can be linearized by using the first-order terms of a Taylor expansion:

$$f(x,y) \cong f(\bar{x},\bar{y}) + \left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}} (y-\bar{y}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y}} (x-\bar{x}) \tag{2.2-2}$$

where the reference point for linearization is the normal steady-state operating point $(\bar{x},\bar{y})$.

By definition, at the normal steady-state operating point,

$$(y - \bar{y}) = 0, \quad (x - \bar{x}) = 0, \quad \text{and} \quad \frac{dy}{dt} = f(y,x) = 0$$

Therefore, $f(\bar{y},\bar{x}) = 0$.

Definitions of "deviation variables" are automatically produced by the expansion:

$$y^{\Delta} = (y - \bar{y}) \quad \text{and} \quad x^{\Delta} = (x - \bar{x}) \tag{2.2-3}$$

Here, a delta is used to denote the deviation variables, rather than the prime used by Seborg, et al., in order to avoid confusion with the notation used here for first derivatives.

Since

$$\frac{dy^{\Delta}}{dt} = \frac{dy}{dt} \tag{2.2-4}$$

The linearized form of Eqn. (2.2-1) becomes:

$$\frac{dy^{\Delta}}{dt} = \left.\frac{\partial f}{\partial y}\right|_s y^{\Delta} + \left.\frac{\partial f}{\partial x}\right|_s x^{\Delta} \tag{2.2-5}$$

where the normal steady-state operating point is now denoted by "s" rather than $(\bar{x}, \bar{y})$. See Seborg, et al., p. 87, for cases which involve additional input variables.

An example is:

$$\frac{dy}{dt} = a_1 y^2 + a_2 xy + a_3 \frac{x}{y} + a_4 x^2 \tag{2.2-6}$$

$$\frac{dy^{\Delta}}{dt} = \left(2 a_1 \bar{y} + a_2 \bar{x} - a_3 \frac{\bar{x}}{\bar{y}^2}\right) y^{\Delta} + \left(a_2 \bar{y} + \frac{a_3}{\bar{y}} + 2 a_4 \bar{x}\right) x^{\Delta} \tag{2.2-7}$$

where $a_1$ - $a_4$ are constants. Systems of more than one differential equation can also be linearized by the same procedure. For a two-equation, three-variable system, for example:

$$\frac{dy}{dt} = f(x,y,z) \tag{2.2-8}$$

$$\frac{dx}{dt} = g(x,y,z) \tag{2.2-9}$$

where x(t) and y(t) are dependent variables and z(t) is an independent input variable. The linearized forms are:

$$\frac{dy^{\Delta}}{dt} = \left.\frac{\partial f}{\partial y}\right|_s y^{\Delta} + \left.\frac{\partial f}{\partial x}\right|_s x^{\Delta} + \left.\frac{\partial f}{\partial z}\right|_s z^{\Delta} \tag{2.2-10}$$

$$\frac{dx^{\Delta}}{dt} = \left.\frac{\partial g}{\partial y}\right|_s y^{\Delta} + \left.\frac{\partial g}{\partial x}\right|_s x^{\Delta} + \left.\frac{\partial g}{\partial z}\right|_s z^{\Delta} \tag{2.2-11}$$

## 2.3. Dimensionless and Deviation Variables

In the analysis of simple dynamic systems, you may be able to work with your balance equations in terms of the original variables. In most cases, however, you will want to convert the variables into either dimensionless variables, deviation variables, or dimensionless deviation variables.

Making a system of equations dimensionless

- simplifies the equations so you can see their fundamental mathematical forms more easily,
- provides insight by showing you that only dimensionless groups of parameters are important, not individual parameters,
- often scales variables so they have values between 0 and ± 1.

Defining deviation or dimensionless deviation variables allows you to easily see when a system is running at the desired operating conditions and when it is deviating from these conditions.

Dimensionless variables:

A general definition of dimensionless variables is:

$$y^* = \frac{y}{y_c} \text{ or } \frac{y}{(g)_c} \text{ or } \frac{y - y_c}{y_c} \text{ or } \frac{y - y_{c1}}{y_{c2} - y_{c1}} \text{ etc.} \qquad (2.3\text{-}1)$$

where

$y_c$, $y_{c1}$, and $y_{c2}$ are characteristic constant values of the variable y(t), and where $y_{c1}$ and $y_{c2}$ are often the minimum and maximum values, respectively, of y(t) that will be encountered,

$(g)_c$ is a characteristic constant value of a parameter group with the same dimensions as y(t).

When making time dimensionless, the constant value of a parameter group with the dimension of time is used. Select this constant parameter group in order to achieve the maximum simplification of the form of the most important differential equation in the system. For example, one form of the energy balance for an electrically heated stirred tank with a "fast response" heating element is:

$$\frac{dT}{dt} + \left(\frac{w}{\rho V} + \frac{UA}{\rho CV}\right) T = \left(\frac{w}{\rho V}\right) T_i + \left(\frac{1}{\rho CV}\right) Q + \left(\frac{UA}{\rho CV}\right) T_{amb} \qquad (2.3\text{-}2)$$

Note that the dimension of the group (w/ρV + UA/ρCV) is (1/time). Thus, the inverse of this group is a constant, ρ or $t_c$, that can be used to define a dimensionless time, $t^*$, which results in a simple form of the equation:

$$\frac{dT}{dt^*} + T = \left(\frac{\tau w}{\rho V}\right) T_i + \left(\frac{\tau}{\rho CV}\right) Q + \left(\frac{\tau UA}{\rho CV}\right) T_{amb} \qquad (2.3\text{-}3)$$

where

$$t^* = \frac{t}{\tau} \text{ where } \qquad \tau = \frac{1}{\left(\frac{w}{\rho V} + \frac{UA}{\rho CV}\right)} = \text{a constant value}$$

Be careful to check the definition of symbols when reading equations. Often the dimensionless time is labeled t, rather than the characteristic time constant.

Deviation Variables:

A general definition of deviation variables is:

$$y^{\Delta} = y - \bar{y} \tag{2.3-4}$$

where $\bar{y}$ is the constant value of y (t) at the "normal steady-state operating conditions." These are the conditions that the system usually runs at, or is designed to run at, or that you want to control the system to run at. Note that the deviation variable here is not dimensionless and has the same dimensions as y(t). Note also that deviation variables can take on negative values even when the original variables can't. Explicit definition of a deviation time variable is not usually done since we usually implicitly think of our analysis starting at t = 0 or $t^* = 0$.

Deviation variables are often used when considering the dynamic behavior of processes and the control of processes. If all the values of the deviation variables in a particular system are at zero, you know immediately that the system is running where it should.

Note that deviation variables are automatically defined when linearizing a system of nonlinear equations. Deviation variables can also be defined for a system of equations which are already linear.

Deviation variables are especially useful when using Laplace transforms to analyze a system. The following are the Laplace transforms of the first and second derivatives, with respect to time, of the variable y(t):

$$\mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0) \tag{2.3-5}$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) \tag{2.3-6}$$

We usually consider systems "initially at rest," that is, systems that are at their normal steady-state operating conditions at t = 0. So, when y(t) is defined as a deviation variable, y(0) = 0. Since, for a system initially at rest, y'(0) = 0, y''(0) = 0, and so forth for higher derivatives, the above expressions become:

$$L[y^{\Delta}'] = sL[y^{\Delta}] \tag{2.3-7}$$

$$L[y^{\Delta}''] = s^2L[y^{\Delta}] \tag{2.3-8}$$

Thus, when deviation variables are used, an "s" in a transfer function is a "marker" for a first derivative and an "$s^2$" is a marker for a second derivative, etc. Laplace transforms are discussed in detail below.

Dimensionless Deviation Variables:

A general definition of dimensionless deviation variables is:

$$y^{\Delta}* = y* - \bar{y}* = \frac{y - \bar{y}}{y_c} \text{ or } \frac{y - \bar{y}}{(g)_c} \text{ or } \frac{y - \bar{y}}{y_{c2} - y_{c1}} \text{ etc.} \tag{2.3-9}$$

where

$\bar{y}^*$ is the constant value of $y^*(t)$ at the "normal steady-state operating conditions,"

$\bar{y}$ is the constant value of y (t) at the "normal steady-state operating conditions,"

$y_c$, $y_{c1}$, and $y_{c2}$ are constant values of y(t), and where $y_{c1}$ and $y_{c2}$ are often the constant minimum and maximum values, respectively, of y(t) that will be encountered

$(g)_c$ is a characteristic constant value of a parameter group with the same dimensions as y(t).

Note that dimensionless deviation variables can take on negative values even when the original variables can't. Explicit definition of a deviation time variable is not usually done since we usually implicitly think of our analysis starting at $t^* = 0$.

The same simplification of Laplace transforms discussed above for deviation variables will also be obtained for dimensionless deviation variables.

Note that dimensionless deviation variables are automatically defined when linearizing a system of nonlinear dimensionless equations. Deviation variables can also be defined for a system of dimensionless equations which are already linear.

Notes on Procedure:

From the definitions given above, you can see that there will be several options for defining each dimensionless variable or dimensionless deviation variable. For each variable, list the options that you can think of. Select one definition for each variable using the following criteria:

- The definition simplifies the form of the most important equations in the system. This criterion is especially useful when selecting a constant parameter group for making time dimensionless; see the example given above.

- The definition makes physical sense, that is, you can describe it qualitatively and simply in words.

- The definitions scale variables other than time such that they vary between ± 1.

Take the selected definitions and rearrange them so that you have an equation expressing what each variable is equal to in terms of the new variable and the constants involved. For example,

$$y = y^* y_c + y_c$$

$$y = y^* (g)_c$$

$$y = y^{\Delta} + \bar{y}$$

$$y = y^{\Delta^*}(g)_c + \bar{y} \text{ , etc.} \qquad (2.3\text{-}10)$$

Then plug the definition for each of the original variables into the original equations and simplify. You may want to also try other options for defining one or more of the new dimensionless variables.

After you rearrange the resulting equations into a simple form, you will note that there are groups of the original constant parameters and these groups are dimensionless. Write out a qualitative "word definition" of what each dimensionless parameter group represents. For a list of dimensionless parameter groups (or "dimensionless parameters," "dimensionless groups" or "dimensionless numbers") see D. F. Boucher and G. E. Alves, "Dimensionless Numbers," *Chem. Eng. Progr.* **55**(9), 55-64 (1959).

## 2.4. Integrating Factor Approach for First-Order Linear Equations

Consider a physical system, such as a heated stirred tank or a stirred tank with a non-reacting chemical in the feed, which can be described by the general form of a first-order linear differential equation with constant coefficient:

$$\frac{dy(t)}{dt} + ay(t) = f(t)$$

or

$$y' + ay = f \qquad (2.4\text{-}1)$$

The initial condition specified here is y(0) = 0. This is the initial condition for a system that is referred to as "initially at rest". For the heated stirred tank, y(t) represents the deviation of the fluid temperature from the initial (e.g., steady-state) temperature. For a stirred tank with an inert dye in the feed, y(t) represents the deviation of the dye concentration from the initial (e.g., steady-state) concentration. The function f(t) is called the "nonhomogeneous term" of the equation and also the "forcing function" of the physical system represented by Eqn. (2.4-1). For the heated stirred tank, f(t) is associated with the energy input with the inlet fluid and through the heating element. For the stirred tank with an inert dye in the feed, f(t) is associated with the dye in the inlet fluid.

Our goal is to solve Eqn. (2.4-1) to find y(t) for a given parameter "a" and a given f(t).

The homogeneous form of this equation is

$$y_h' + ay_h = 0 \qquad (2.4\text{-}2)$$

with the initial condition specified here to be $y_h(0) = 1$. The solution to this homogeneous equation is

$$y_h(t) = e^{-at} \qquad (2.4\text{-}3)$$

and is represented below as $y_h(t)$ or $y_h$.

We can see that in order to solve Eqn. (2.4-1), we will need to integrate it. The "integrating factor" approach to solving this type of equation involves dividing Eqn. (2.4-1) through by $y_h$ before integrating:

$$\frac{y'}{y_h} + \frac{ay}{y_h} = \frac{f}{y_h} \tag{2.4-4}$$

$$\frac{y'}{y_h} + \frac{ay}{y_h} = \frac{d}{dt}\left[\frac{y}{y_h}\right] = \frac{f}{y_h} \tag{2.4-5}$$

$$y(t) = \frac{y(0)}{y_h(0)} y_h(t) + y_h(t) \int_0^t \frac{f(z)}{y_h(z)} dz \tag{2.4-6}$$

where "z" is the dummy integration variable for "t" in the integral. For the initial conditions we have specified here, $y(0) = 0$ and $y_h(0) = 1$, this solution is

$$y(t) = y_h(t) \int_0^t \frac{f(z)}{y_h(z)} dz \tag{2.4-7}$$

We have found the solution to the general first-order linear differential equation with constant coefficient, however, the integrating factor approach won't work for higher order equations. To solve higher-order equations, we will learn about Laplace transforms.

For the specific initial conditions and Eqn. (2.4-1) given here, the homogeneous solution is $y_h(t) = e^{-at}$. Therefore, Eqn. (2.4-7) is also equal to

$$y(t) = \int_0^t f(z)\, y_h(t-z)\, dz = \int_0^t y_h(z)\, f(t-z)\, dz \tag{2.4-8}$$

The integral in Eqn. (2.4-8) is called a "convolution integral." Eqn. (2.4-8) shows that the solution y(t) is equal to $y_h(t)$ convoluted with f(t). This is also represented as

$$y(t) = f(t) * y_h(t) = y_h(t) * f(t) \tag{2.4-9}$$

where the symbol "$*$", the "convolution operator", represents convolution and not simple multiplication.

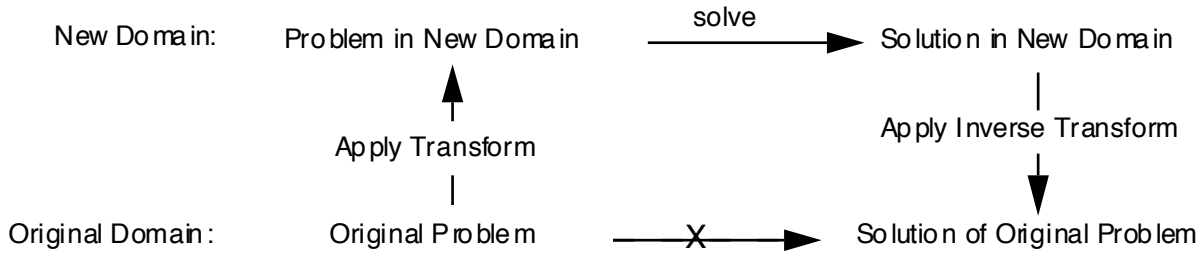
**In general, for two functions g(t) and h(t), convolution is defined by:**

$$g(t) * h(t) = \int_0^t g(z)\, h(t-z)\, dz = \int_0^t h(z)\, g(t-z)\, dz = h(t) * g(t) \tag{2.4-10}$$

Convolution is an important concept and we will use it later in the general solution of linear systems.

## 2.5. Laplace Transform Approach for First- and Higher-Order Equations

Often during the course of solving problems, a "transformation" of the problem from its original "domain" into a new domain makes the solution of the problem easier. The procedure is to transform the problem into the new domain, solve the problem in the new domain, and then "inverse transform" or "back transform" the solution in the new domain into a solution in the original domain of the problem. In a sense, you do this every time you solve an engineering problem: you transform a statement of the problem written or spoken in words into a mathematical model, solve the mathematical model, and then describe the results of the mathematical solution in words.




Solution of a complex system of linear differential equations by analytical integration can be difficult. In this section, we will learn that such a complex system can be solved more easily by first transforming the system of linear differential equations into a system of algebraic equations in a new domain, solving the algebraic equations in the new domain, and then back-transforming to get the final solution in the original domain.

The integrating factor approach can be considered as involving a "transformation" of our starting Eqn. (2.4-1), which is the general form of first-order linear differential equations with constant coefficient. The "integrating factor transform" $\Upsilon$ is defined by

$$\Upsilon_{[y(t)]} = \int_0^t \frac{y(t)}{y_h(t)}\,dt \tag{2.5-1}$$

with the initial conditions $y(0) = 0$ and $y_h(0) = 1$. That is, the transform $\Upsilon$ involves dividing by $y_h$ and integrating from 0 to t. The transform $\Upsilon$ applied to Eqn. (2.4-1) is

$$\Upsilon_{[y' + ay = f]} = \int_0^t \frac{y'(t)}{y_h(t)}\,dt + \int_0^t \frac{ay(t)}{y_h(t)}\,dt = \int_0^t \frac{f(t)}{y_h(t)}\,dt \tag{2.5-2}$$

Since $y_h(t) = e^{-at}$, the $\Upsilon$ transform can also be written as

$$\Upsilon[y(t)] = \int_0^t y(t)\ e^{at}\ dt \qquad (2.5\text{-}3)$$

This transform allows us to solve first-order linear differential equations. However, this approach doesn't work, in general, for second and higher order equations.

For solution of complex linear differential equations, a different transform, the Laplace transform, L, provides a more powerful approach. The Laplace transform of the arbitrary function g(t) is defined by

$$L[g(t)] = \int_0^{\square} g(t)\ e^{st}\ dt \qquad (2.5\text{-}4)$$

NOTE TYPO: this is an integral over all time from t = 0 to infinity, where the box at the upper limit of integration in the equation here and below should be $\infty$.

Just as the integrating factor approach introduced above can be considered a "transform", $e^{-st}$ in the Laplace transform can be considered an "integrating factor."

The Laplace transformation of a linear differential equation essentially involves integrations that remove derivatives from the equation and converts the problem into the solution of algebraic equation (followed finally by an inverse transformation). For a linear differential equation involving the dependent variable y(t), the algebraic equation resulting from application of the Laplace transform to the differential equation is solved for L[y(t)]. Then the inverse transform is applied to the algebraic equation in order to obtain the solution y(t).

**"The Laplace Transform converts a system of linear differential equations with constant coefficients into a system of algebraic equations."**

It is interesting to try to imagine what Laplace's thought process was when he invented the Laplace transform. Let's imagine that he was first aware that the integrating factor for first-order linear equations removed the derivative by multiplying the original equation by a function and then integrating. His goal then was to determine if there was some other function and integration limits that would remove both first and higher-order derivatives from differential equations.

The critical step for Laplace to accomplish was to find a transform that would convert the derivative of an arbitrary function, g'(t), into an expression in terms of the transform of g(t). This same transform, through a step-wise process, would convert g"(t) into an expression in terms of the transform of g'(t), which then could be expressed in terms of the transform of g(t). Thus, all derivatives of a function or variable could be converted into a transform of the original function or variable.

Laplace undoubtedly tried many things. In one of his trials, imagine that he multiplied the function he wanted to transform, g(t), by an unknown function u(t) and took the derivative of their product:

$$\frac{d}{dt}(ug) = u\frac{dg}{dt} + g\frac{du}{dt} = ug' + gu' \tag{2.5-5}$$

We know this derivative from our math tables, so we can take over from Laplace from here. Now integrate from some currently unknown lower limit, LL, to some currently unknown upper limit, UL.

$$ug\Big|_{LL}^{UL} = \int_{LL}^{UL} ug'dt + \int_{LL}^{UL} gu'dt \tag{2.5-6}$$

Rearranging,

$$\int_{LL}^{UL} ug'dt = ug\Big|_{LL}^{UL} - \int_{LL}^{UL} gu'dt \tag{2.5-7}$$

We can see that the LHS of the equation will define our transform of g'(t) since the RHS of the equation does not involve g'(t), showing that we can successfully remove g'(t) from our problem. We are definitely on the right track. All we have to do now is to pick a "convenient" u(t) and "convenient" limits UL and LL. After trying a few things, we find that $u(t) = e^{-st}$, where "s" is a constant with units of ($time^{-1}$, or "frequency"), $UL = \infty$, and LL = 0 are "convenient" since:

(a) $ug = 0$ at $UL = \infty$, since $u(\infty) = e^{-s(inf)} = 0$, (2.5-8)

(b) $ug = g(0)$ at LL = 0, since $u(0) = e^{-s0} = 1$, and (2.5-9)

(c) $u' = -s\,e^{-st}$, so that (2.5-10)

$$\int_{LL}^{UL} gu'dt = -s\int_{LL}^{UL} gu\,dt \tag{2.5-11}$$

So, we define our "reinvented" Laplace transform, Eqn. (2.5-4):

$$\mathcal{L}[g(t)] = \int_0 g(t)\,e^{-st}\,dt \tag{2.5-4}$$

with it's property of "eliminating" y'(t) from solutions:

$$\mathcal{L}[y'(t)] = \int_0 y'(t)\,e^{-st}\,dt = s\int_{LL}^{UL} y(t)\,e^{-st}\,dt - y(0) = s\,\mathcal{L}[y(t)] - y(0) \tag{2.5-12}$$

that is,

$$\mathcal{L}[y'(t)] = s\,\mathcal{L}[y(t)] - y(0) \qquad (2.5\text{-}13)$$

**The new domain that the Laplace transform converts systems into is called the "Laplace domain," the "s domain," or the "frequency domain," since the dimension of s is (1/time) or (frequency). The original domain is the "time domain."**

The real power of the Laplace transform is that it can be applied successively in order to reduce second-order and higher derivatives of a function down to transforms of the function. For example, for a second derivative,

$$\mathcal{L}[y''(t)] = s^2\,\mathcal{L}[y(t)] - s\,y(0) - y'(0) \qquad (2.5\text{-}14)$$

This is derived in Eqns. (3-10) through (3-13) on page 45 of Seborg, et al.

We usually use deviation variables and consider systems "initially at rest," that is, systems that are at their normal steady-state operating conditions at t = 0. In this case, y(0) = 0, y'(0) = 0, y''(0) = 0, and so forth for higher derivatives. The transforms of derivatives then simplify to:

$$\mathcal{L}[y'] = s\,\mathcal{L}[y] \quad \text{for deviation variable and system initially at rest} \qquad (2.5\text{-}15)$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] \quad \text{for deviation variable and system initially at rest} \qquad (2.5\text{-}16)$$

and so forth for higher derivatives. An "s" in a transform is a "marker" for a first derivative, an "$s^2$" is a marker for a second derivative, etc., for deviation variables and systems initially at rest.

Now we can apply the Laplace transform approach to the solution to our Eqn. (2.4-1),

$$\mathcal{L}[y' + ay = f] = s\,\mathcal{L}[y] - y(0) + a\,\mathcal{L}[y] = \mathcal{L}[f] \qquad (2.5\text{-}17)$$

We usually consider systems "initially at rest," so, for the initial condition y(0) = 0,

$$\mathcal{L}[y] = \left(\frac{1}{s+a}\right)\mathcal{L}[f] \qquad (2.5\text{-}18)$$

This can also be written in equivalent notation as

$$Y(s) = \left(\frac{1}{s+a}\right)F(s) \qquad (2.5\text{-}19)$$

**The terms L[y] = L[y(t)] = L[y(t)](s) = Y(s) = Y are equivalent.**

We get our solution y(t) by taking the inverse transform

$$y = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\left(\frac{1}{s+a}\right)F(s)\right] \qquad (2.5\text{-}20)$$

Derive for yourself that the transform of the homogeneous form of this differential equation with $y_h(0) = 1$ is

$$Y_h(s) = \left(\frac{1}{s+a}\right) \tag{2.5-21}$$

Thus, the transform solution, Eqn. (2.5-19), of the original equation (2.4-1) can be written

$$Y(s) = Y_h(s) \cdot F(s) \tag{2.5-22}$$

and

$$y(t) = y_h(t) * f(t) \tag{2.5-23}$$

Eqn. (2.5-23) is the same as Eqn. (2.4-9). We have reached the same solution using Laplace transforms that we did using the integrating factor approach.

**Note that the product of the transforms of two functions is equal to the transform of the convolution of the two functions:**

$$Y_h(s) \cdot F(s) = \mathcal{L}[y_h] \cdot \mathcal{L}[f] = \mathcal{L}[y_h * f] \tag{2.5-23}$$

**In general, for two arbitrary functions g(t) and h(t):**

$$G(s) \cdot H(s) = H(s) \cdot G(s) = \mathcal{L}[h] \cdot \mathcal{L}[g] = \mathcal{L}[h * g] = \mathcal{L}[g * h] \tag{2.5-24}$$

**Be careful,**

$$L[h] \cdot L[g] \neq L[h \cdot g] \tag{2.5-25}$$

## 2.6. Responses to Unit Impulses and Transfer Functions

The Dirac delta funtion is defined here by

$$\delta(t - t_j) = 0 \quad \text{for } t < t_j \tag{2.6-1}$$

$$\delta(t - t_j) = 0 \quad \text{for } t > t_j \tag{2.6-2}$$

$$\int_0 \delta(t - t_j)[1/\text{dimension of time}]\, dt = 1 \text{ [dimensionless]} \tag{2.6-3}$$

A "unit impulse" at $t = t_j$ in the arbitrary input variable or forcing function h(t) is defined as:

$$h(t) = 1[\text{dimensions of } h(t) \bullet \text{dimension of } t] \quad \delta(t - t_j)[1/\text{dimension of } t] \tag{2.6-4}$$

$$\int_0 h(t)dt = \int_0 1\,[\text{dimen. of } h(t) \bullet \text{dimen. of } t] \qquad \delta_{(t-t_j)}\,dt = 1\,[\text{dimen. of } h(t) \bullet \text{dimen. of } t] \tag{2.6-5}$$

where "unit" refers to the integral being equal to one. The "magnitude" of an impulse refers to the value of the integral of the impulse over time.

**The response of an output variable to a unit impulse in an input variable is a key feature of the dynamic behavior of linear systems.**

The Laplace transform of a unit impulse at $t = t_j$ in h(t) is:

$$H(s) = \mathcal{L}[h(t)] = \mathcal{L}[\delta_{(t-t_j)}] = e^{-s\,t_j}, \text{ for } h(t) = \delta_{(t-t_j)} \tag{2.6-6}$$

Note that $e^{-s\,t_j}$ is the Laplace transform of a pure time delay of time $t_j$. The Laplace transform of a unit impulse at $t = t_j = 0$ is:

$$H(s) = \mathcal{L}[h(t)] = \mathcal{L}[\delta_{(t)}] = 1, \text{ for } h(t) = \delta_{(t)} \tag{2.6-7}$$

The Laplace solution for a unit impulse at t = 0 in f(t) for the physical system described by Eqn. (2.4-1) is:

$$\frac{dy(t)}{dt} + a\,y(t) = f(t) = \delta_{(t)} \tag{2.6-8}$$

$$s\,Y(s) + a\,Y(s) = 1 \quad \text{for the deviation variable } y(t) \text{ and system initially at rest} \tag{2.6-9}$$

$$Y(s) = \left[\frac{1}{s+a}\right] = \text{transform of response of } y(t) \text{ to a unit impulse in the entire forcing function } f(t) \tag{2.6-10}$$

$$y(t) = e^{-at} = \text{response of } y(t) \text{ to a unit impulse in the entire forcing function } f(t) \tag{2.6-11}$$

From Eqn. (2.4-3), we know that Eqn. (2.6-11) is also the homogeneous solution of the system equation with $y_h(0) = 1$ and that Eqn. (2.6-10) is the Laplace transform of this homogeneous solution. We emphasize "unit impulse in the entire forcing function f(t)" in Eqns. (2.6-10) and (2.6-11) because the homogeneous solution of a system equation equals the response to a unit impulse in the entire forcing function not, in general, to unit impulses in separate input variables which may be a part of the complete forcing function. This point is discussed below and does not affect the general conclusion we reach here.

Returning to our original equation for this case, Eqn. (2.4-1), and taking the Laplace transform of the entire equation,

$$s\,Y(s) + a\,Y(s) = F(s) \tag{2.6-12}$$

Thus we find that for an arbitrary input f(t):

$$Y(s) = \left[\frac{1}{s+a}\right] F(s) = G(s)\,F(s) \tag{2.6-13}$$

$$y(t) = e^{-at} * f(t) = g(t) * f(t) \tag{2.6-14}$$

By comparing Eqn. (2.6-11) to Eqn. (2.6-14), we see that the function g(t) is the response of y(t) to a unit impulse in f(t). From Eqn. (2.6-14), we can conclude that the output y(t) of the system, which is initially at rest, is given by the response of y(t) to a unit impulse in f(t) convoluted with the arbitrary input f(t).

Whereas it is possible to consider a unit impulse of the entire forcing function of a system equation, we almost always consider the unit impulse of each of the input variables separately. For example, for our simple first-order system,

$$f(t) = b_1 x(t) + b_2 z(t), \quad \text{where} \quad b_1 \text{ and } b_2 \text{ are constants} \tag{2.6-15}$$

$$\frac{dy(t)}{dt} + a y(t) = f(t) = b_1 x(t) + b_2 z(t) \tag{2.6-16}$$

A unit impulse input in x(t) is

$$x(t) = \delta(t), \quad \int_0 x(t)\,dt = \int_0 \delta(t)\,dt = 1 \text{ [dimensions of } x(t) \bullet \text{dimension of } t\text{], and } z(t) = 0 \tag{2.6-17}$$

In this case, for a unit impulse input in x(t):

$$s\,Y(s) + a\,Y(s) = b_1 \tag{2.6-18}$$

$$Y(s) = \left[\frac{b_1}{s+a}\right] = \text{transform of response of y(t) to a unit impulse in x(t)} \tag{2.6-19}$$

$$y(t) = b_1 e^{-at} = \text{response of y(t) to a unit impulse in x(t)} \tag{2.6-20}$$

Similarly,

$$Y(s) = \left[\frac{b_2}{s+a}\right] = \text{transform of response of y(t) to a unit impulse in z(t)} \tag{2.6-21}$$

$$y(t) = b_2 e^{-at} = \text{response of y(t) to a unit impulse in z(t)} \tag{2.6-22}$$

Returning to our original equation for this case, Eqn. (2.6-16),

$$\frac{dy(t)}{dt} + ay(t) = b_1 x(t) + b_2 z(t) \qquad (2.6\text{-}23)$$

The Laplace transform of this equation is:

$$s\,Y(s) + a\,Y(s) = b_1\,X(s) + b_2\,Z(s) \qquad (2.6\text{-}24)$$

Thus we find that for arbitrary inputs in x(t) and z(t):

$$Y(s) = \left[\frac{b_1}{s+a}\right] X(s) + \left[\frac{b_2}{s+a}\right] Z(s) = G_1(s)\,X(s) + G_2(s)\,Z(s) \qquad (2.6\text{-}25)$$

$$y(t) = b_1\, e^{-at} * x(t) + b_2\, e^{-at} * z(t) = g_1(t) * x(t) + g_2(t) * z(t) \qquad (2.6\text{-}26)$$

By comparing Eqns. (2.6-20) and (2.6-22) to Eqn. (2.6-26), we see that the function $g_1(t)$ is the response of y(t) to a unit impulse in x(t), and $g_2(t)$ is the response of y(t) to a unit impulse in z(t). By comparing Eqns. (2.6-19) and (2.6-21) to Eqn. (2.6-25), and by comparing Eqn. (2.6-25) to Eqn. (2.6-26), we see that the Laplace transform of the "unit impulse response" of y(t) to x(t), $g_1(t)$, is $G_1(s)$, and the Laplace transform of the unit impulse response of y(t) to z(t), $g_2(t)$, is $G_2(s)$. $G_1(s)$ is called the **TRANSFER FUNCTION** relating y(t) and x(t). Similarly, $G_2(s)$ is the transfer function relating y(t) and z(t).

Our **GENERAL CONCLUSIONS** for linear systems initially at rest using deviation variables are:

**The response of an output variable to a given input variable is equal to the response of the output variable to a unit impulse in the input variable convoluted with the input.**

**The transfer function relating a pair of input and output variables is equal to the Laplace transform of the response of the output variable to a unit impulse in the input.**

**The Laplace transform of the response of an output variable is equal to the transfer function relating that pair of input and output variables multiplied times the Laplace transform of the input.**

These statements are true for all linear systems with constant coefficients, even those with system equations and forcing functions much more complex than Eqn. (2.6-16), including higher order equations and forcing functions with derivatives of input variables.

We can apply these concepts to subunits (blocks in a block diagram showing information flow) of complex linear systems, in addition to applying it to the entire system.

## 2.7. Response to a Series of Impulses

In general, our problem is that we wish to find the response of an output variable for any arbitrary variation in an input variable. We have already found the general solution to this

problem in Section 2.6. Another approach to reaching this solution, which is somewhat more intuitive or "visual," starts by breaking a given input forcing function into a series of many Dirac deltas or impulses. The magnitudes of the impulses would vary such that the summation of the magnitudes of the impulses equals the integral of the input forcing function over time.

Consider a stirred tank with a liquid stream of constant flow rate q(liter/s) going in and out such that the fluid volume in the tank, V(liter), remains constant. You may find it useful to imagine a clear glass tank and pipes with clear water flowing through them. A inert dye (inert tracer) is be present in varying concentration, $c_i$ (mol/liter), in the inlet stream.

$$V\frac{dc}{dt} = q\,c_i - qc \tag{2.7-1}$$

or

$$\tau\frac{dc}{dt} + c = c_i \quad \text{where the "time constant"} \quad \tau_{(s)} = \frac{V}{q} \tag{2.7-2}$$

or

$$\frac{dc}{dt} + \left(\frac{1}{\tau}\right)c = \left(\frac{1}{\tau}\right)c_i \tag{2.7-3}$$

For this "mixing tank" system, we also could define a deviation concentration variable and/or make the concentration and time variables dimensionless. Compare Eqn. (2.7-3) to Eqns. (2.6-8) and (2.4-1). They are equivalent when y(t) = c(t), a = $b_1$ = (1/τ), and z(t) = 0. Also note the similarities between Eqns. (2.7-3), (2.6-8), and (2.4-1) to the dimensionless Eqn. (1-1) for the simple heated tank.

An impulse input to the mixing tank can be approximated in "real life" by a short rectangular pulse or "slug" of concentrated dye in the inlet stream. To approximate an impulse, the duration of the slug, Δt[s], must be short with respect to the characteristic time constant of the system, which in this case is equal to τ[s]. To approximate a unit impulse input, the slug would have a concentration $c_i$ = 1[mol•s/liter]/Δt[s]. At the flow rate q[liter/s] for Δt[s], the slug would contain q moles of dye, producing c(0) = (1/τ)[mol/liter]. A unit impulse input could also be approximated by rapidly dumping a beaker containing q moles of dye into the tank.

Now take the system "initially at rest" where, before t = 0, the concentrations of dye in the inlet liquid stream and in the tank are zero. At t = 0, a short slug of dye in the inlet stream, approximating an impulse input of magnitude (value of integral over time in [mol·s/liter]) = $c_0$[mol·s/liter], enters the tank. The system equation is

$$\tau\frac{dc(t)}{dt} + c(t) = c_i(t) = c_0\,\delta_{(t-0)} \tag{2.7-4}$$

where

$$c(t<0) = 0 \tag{2.7-5}$$

and where the coefficient $c_0$ has the dimensions [mol·s/liter]. After t = 0 there is no further input of dye into the tank since $c_i(t > 0) = 0$. A unit impulse input in $c_i$ at t = 0 $\{\int c_i(t)dt = \int \delta(t)dt = 1$ [mol·s/liter]$\}$ produces an initial concentration $c(0) = (1/\tau)$[mol/liter], as we calculated above for a slug approximating a unit impulse. Therefore, Eqn. (2.7-4) describing the system is equivalent to

$$\text{for } t \geq 0,\ \tau\frac{dc}{dt} + c = 0 \quad \text{I.C.} \quad c(0) = (c_0/\tau)\ \text{[mol/liter]} \tag{2.7-6}$$

The response c(t) for a this impulse of magnitude $c_0$ in $c_i(t)$, is:

$$c = (c_0/\tau)\, e^{-t/\tau} = (c_0/\tau)\, e^{-(t-0)/\tau} \tag{2.7-7}$$

For the same system, initially at rest, when a separate impulse input of magnitude $c_1$ is made at $t_1 > 0$

$$c = 0 \quad \text{for } t < t_1 \tag{2.7-8}$$

$$c_i = c_1\,\delta(t - t_1) \tag{2.7-9}$$

$$c = (c_1/\tau)\, e^{-(t - t_1)/\tau} \quad \text{for } t \_ \quad t_1 \quad (\text{should be for } t \geq t_1) \tag{2.7-10}$$

or

$$c = (c_1/\tau)\, e^{-(t - t_1)/\tau}\, S(t - t_1) \quad \text{for } t \_ 0 \quad (\text{should be for } t \geq 0) \tag{2.7-11}$$

where $S(t - t_1)$ is the "unit step function" and is defined as = 0 for t < any specified time $t_1$ and = 1 for $t \geq t_1$. For an impulse input of magnitude $c_0$ at t = 0 followed in the same experiment by an impulse input of magnitude $c_1$ at time $t_1$,

$$c_i = c_0\,\delta(t - 0) + c_1\,\delta(t - t_1) \tag{2.7-12}$$

$$c = (c_0/\tau)\, e^{-(t-0)/\tau}\, S(t - 0) + (c_1/\tau)\, e^{-(t - t_1)/\tau}\, S(t - t_1) \tag{2.7-13}$$

Note that the output is just the summation, or "superposition" of the two separate impulse responses. This is a feature common to all linear systems.

The figures on the next two pages show the response of this system to a unit impulse, a series of five consecutive impulse inputs, and the response to a series of five consecutive impulse inputs.

In general, we wish to find the "response" or output c(t) for any arbitrary input $c_i(t)$. We can think of breaking a given $c_i(t)$ into a series of many Dirac deltas or impulses. The magnitudes of the impulses would vary so that the shape of the series of impulses vs. time had the same shape as the specified $c_i(t)$ and so that the summation of the magnitudes of the impulses equals the integral of $c_i(t)$ over time.

Now define $c^*(t)[mol/liter] = 1[mol \cdot s/liter] \cdot g(t)[1/s]$ as being the response of c(t) to the unit impulse $c_i(t)[mol/liter] = 1[mol \cdot s/liter] \cdot \delta(t)[1/s]$, or the "unit impulse response" of c(t) to $c_i(t)$. The function g(t) by itself is also referred to as the unit impulse response of c(t) to $c_i(t)$, since the value of the coefficient with the dimensions of the impulse magnitude is always one for a unit impulse. For this system,

$$g(t)[1/s] = \left(\frac{1}{\tau_{[s]}}\right) e^{-(t - t_1)/\tau} \tag{2.7-14}$$

$$g(t - t_1)\, S(t - t_1) = (1/\tau)\, e^{-(t - t_1)/\tau}\, S(t - t_1) \tag{2.7-15}$$

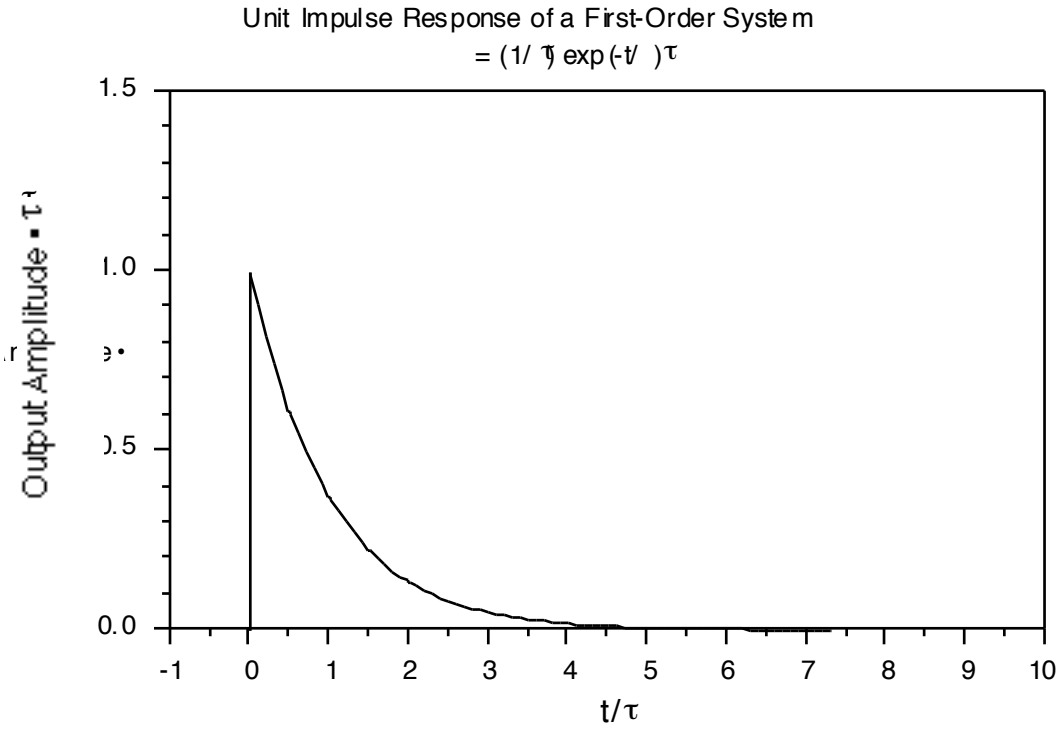
Eqn. (2.7-13) becomes

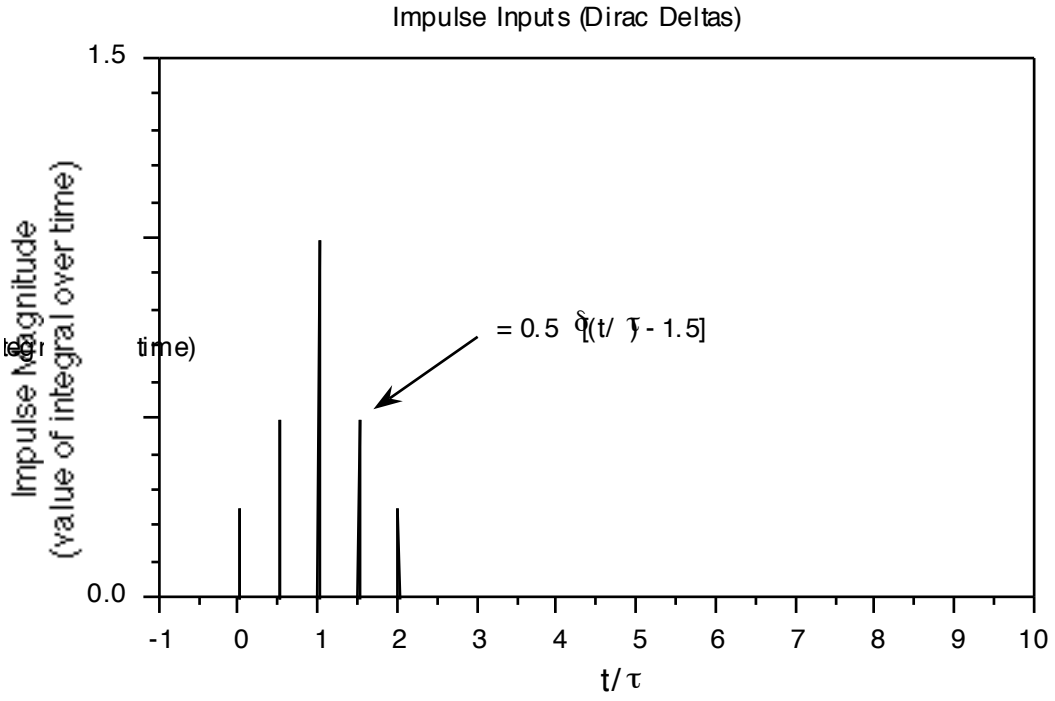
$$c = c_0\, g(t) + c_1\, g(t - t_1)\, S(t - t_1) \tag{2.7-16}$$

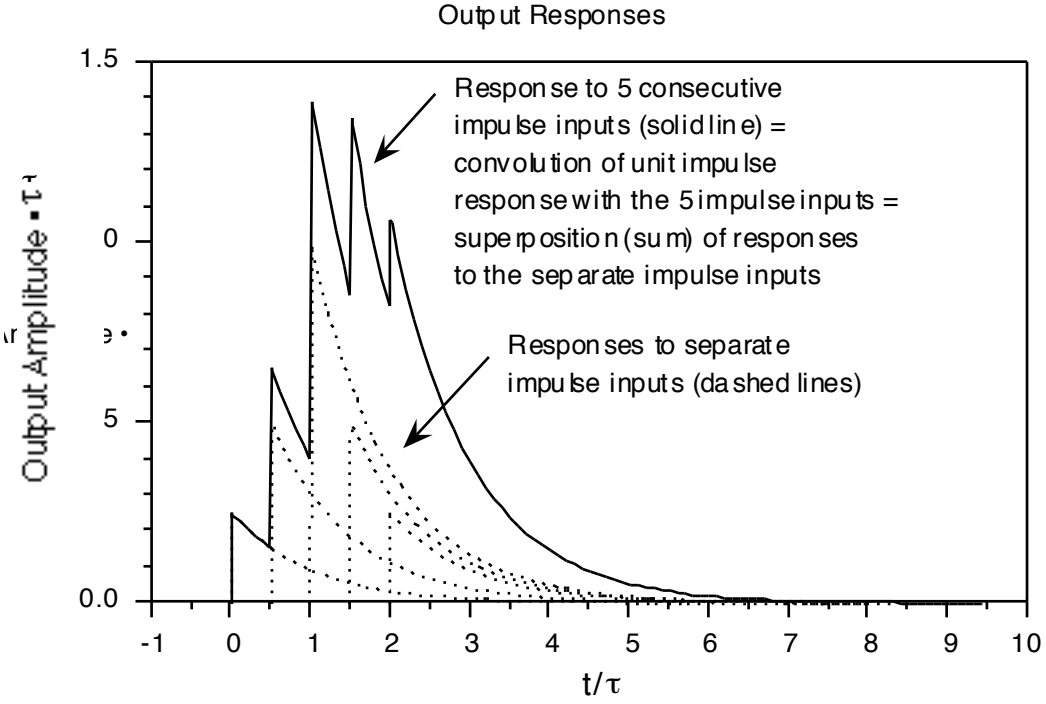
Remember that we want to find c(t) for the arbitrary input $c_i(t)$. Consider an input that starts at t = 0 and ends after $t[s] = t_D[s]$. We can approximate $c_i(t)[mol/liter]$ by a series of N impulses, at times $z_k[s] = kt_D/N$, each of magnitude (value of integral over time in $[mol \cdot s/liter]$) = $\{c_i(z_k)[mol/liter] \cdot \Delta z[s]\}$:

$$c_i(t) = \sum_{k=1}^{N} c_i(z_k)\,\delta(t - z_k)\,\Delta z \tag{2.7-17}$$

where $\Delta z[s] = t_D/N$, the time interval between impulses, and k = 1, 2, ..., N. The coefficients $c_i(z_k)$ in Eqn. (2.7-17) have the values of $c_i(t)$ at the times $t = z_k$.


Unit Impulse Response of a First-Order System
= (1/ τ) exp (-t/ )τ
1.5
1.0
0.5
0.0
-1
0
1
2
3
4
5
6
7
8
9
10
t/τ


Impulse Inputs (Dirac Deltas)
1.5
0.0
Impulse Magnitude
(value of integral over time)
= 0.5 δ[(t/τ) - 1.5]
-1
0
1
2
3
4
5
6
7
8
9
10
t/τ


Output Responses
1.5
0
5
0.0
Output Amplitude • τ
Response to 5 consecutive
impulse inputs (solid line) =
convolution of unit impulse
response with the 5 impulse inputs =
superposition (sum) of responses
to the separate impulse inputs
Responses to separate
impulse inputs (dashed lines)
-1
0
1
2
3
4
5
6
7
8
9
10
t/τ

The response to this series of impulses that approximates our arbitrary $c_i(t)$ is

$$c_i(t) = \sum_{k=1}^{N} c_i(z_k) g(t - z_k) S(t - z_k) \Delta z \tag{2.7-18}$$

In the limit that the time interval $\Delta z$ approaches zero:

$$c(t) \cong \int_0^{t_D} c_i(z) g(t - z) S(t - z) dz = \int_0^{t} c_i(z) g(t - z) dz \tag{2.7-19}$$

for $t > t_D$. This is the response of the concentration in the mixing tank to the series of impulses that approximates the arbitrary inlet concentration $c_i(t)$. The response to a continuous arbitrary $c_i(t)$ is

$$c(t) = \int_0^{t} c_i(z) g(t - z) dz = \int_0^{t} g(z) c_i(t - z) dz \tag{2.7-20}$$

or

$$c(t) = c_i(t) * g(t) = g(t) * c_i(t) \tag{2.7-21}$$

where $*$ is the convolution operator. Thus,

$$C(s) = C_i(s)\, G(s) = G(s)\, C_i(s) \tag{2.7-22}$$

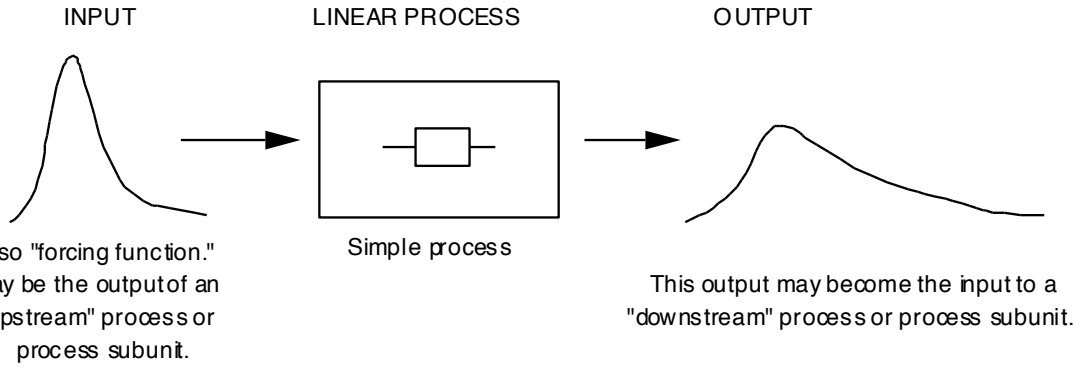
We can see that we have reached the same solution using this approach that we did using the integrating factor approach [compare Eqn. (2.7-21) to Eqn. (2.4-9)] and the Laplace transform approach [compare Eqn. (2.7-21) to Eqn. (2.6-14) and Eqn. (2.7-22) to Eqn. (2.6-13)]. Eqns. (2.7-21) and (2.7-22) apply to an input-output pair in all linear systems, not just the simple one we have used here as an example.

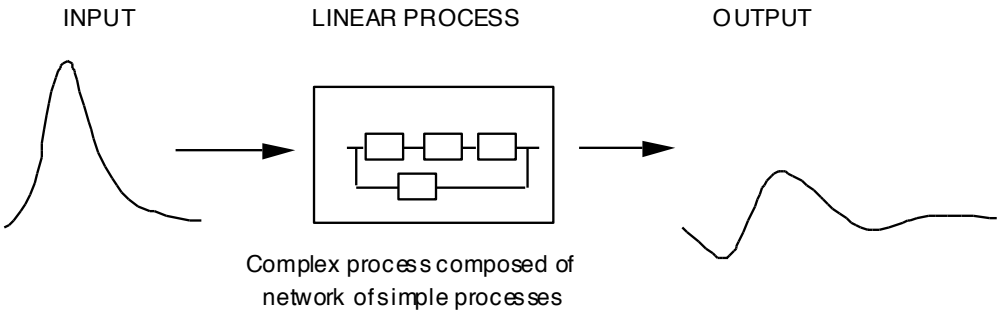
A review of what we have discussed is presented on the following pages.

**Review:**

We consider linear processes (processes that can be described by linear differential equations with constant coefficients) or nonlinear processes that have been linearized about the normal steady-state operating conditions.

There may be several inputs (independent variables) and several outputs (dependent variables). Here, we consider a change in one input variable and the response of one output variable. Processes may be simple or may be complex networks of simple processes:



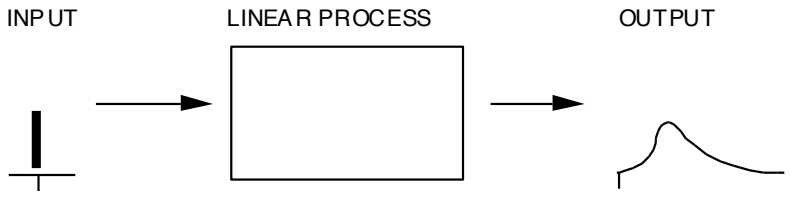




There are three elements to the problem: input, process, output. In general, we will know two elements and have to find the third. Specific objectives will be to:

(1) predict the output, given an input and a process,

(2) design a control system for a process that will give the desired output in response to a given input (e.g., maintain current output, within a specified band, in response to a unit step input),

(3) determine what the internal structure of the process is, given pairs of inputs and outputs.




unit impulse

$x(t) = \delta(t)$

response of output variable to a unit impulse in this input variable, or "unit impulse response" for this pair of input and output variables

Integral of unit impulse over time = 1 with the dimensions of x(t) times the dimension of time. E.g., [mol•s/liter] for x[mol/liter] and t[s], and dimensionless if both x(t) and t are dimensionless. Time duration of "real world" rectangular pulse or gaussian must be << characteristic time (time constant) of process to approx. a $\delta$. Here use height of line to represent magnitude of integral.

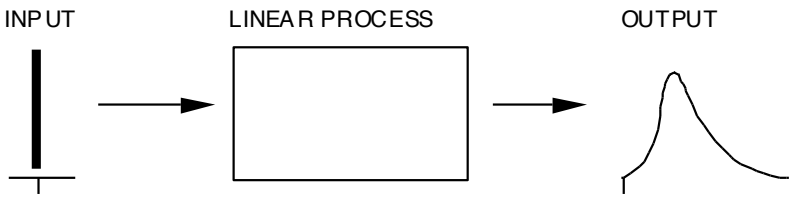
convolution operator

$y(t) = g(t) * x(t) = g(t)$

"unit impulse response"

$L[x(t)] = X(s) = 1$

$Y(s) = L[g(t) * x(t)] = G(s)\,X(s) = G(s)$



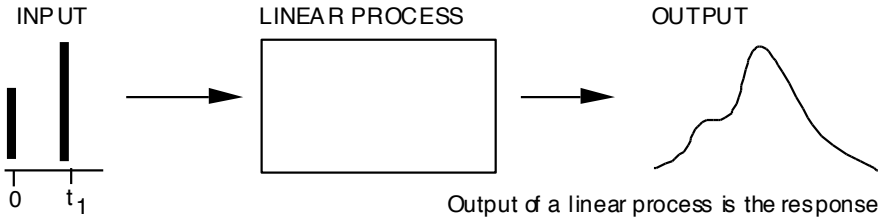

Output of a linear process is the response of the output variable to a unit impulse in the input variable convoluted with the input. Here, output = 2 x "unit impulse response"

$x(t) = 2\ \delta(t)$

$X(s) = 2$

$y(t) = g(t) * x(t) = 2\,g(t)$

$Y(s) = G(s)\,X(s) = 2\,G(s)$




Output of a linear process is the response of the output variable to a unit impulse in the input variable convoluted with the input. Here, the output equals the superposition (sum) of the separate reponses to the two impulse inputs.

$x(t) = \delta(t) + 2\ \delta(t - t_1)$

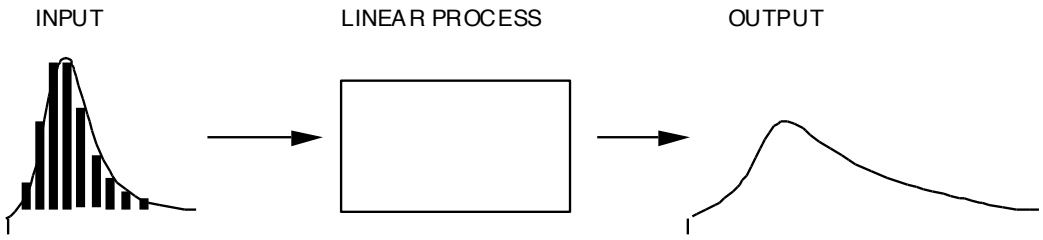
$X(s) = 1 + 2\ e^{-t_1 s}$

Laplace of time delay

unit step function

$y(t) = g(t) * x(t) = g(t) + 2\,S(t - t_1)\,g(t - t_1)$

$Y(s) = G(s)\,X(s) = G(s) + 2\ e^{-t_1 s}\,G(s)$




Any input x(t) of duration $t_D$ can be approximated as series of N impulses, of the different magnitudes $x(z_k)\cdot\Delta z$, at the times $z_k$ and the interval $\Delta z = t_D/N$.

$$x(t) \cong \sum_{k=1}^{N} x(z_k)\,\delta(t - z_k)\,\Delta z$$

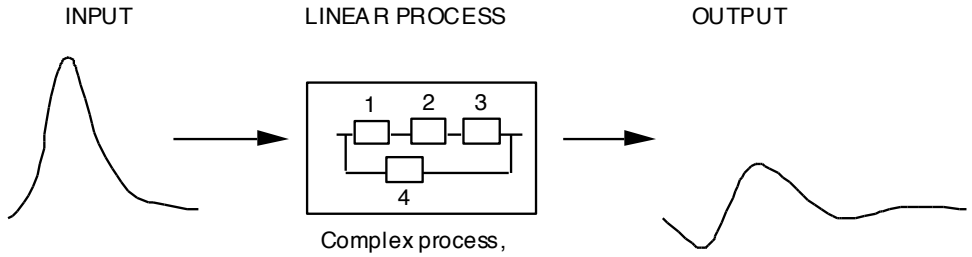
$$X(s) \cong \sum_{k=1}^{N} x(z_k)\,e^{-st_k}\,\Delta z$$

$$y(t) = g(t) * x(t) \cong \sum_{k=1}^{N} x(z_k)\,S(t - z_k)\,g(t - z_k)\,\Delta z$$

$$Y(s) = G(s)\,X(s) \cong \sum_{k=1}^{N} x(z_k)\,e^{-st_k}\,G(s)\,\Delta z$$

The impulse series approximation is shown here to give an intuitive feel for linear process response and convolution. In many cases you will be given an x(t) for which you can find an exact X(s) and Y(s). (If the input is really a impulse series, then the sums shown here are the exact convolutions and transforms for the series.)

G(s) is the Laplace transform of the response of y(t) to a unit impulse in x(t). G(s) is called the TRANSFER FUNCTION relating this pair of input & output variables.




Complex process, here composed of network of subunits (processes)

$x(t)$

$X(s) = \mathcal{L}[x(t)]$

$y(t) = g(t) * x(t)$

$Y(s) = G(s)\,X(s)$

The overall transfer function (or the overall unit impulse reponse) can be determined from the transfer functions (or the unit impulse responses) of the subunits. For the process shown here:

$$G(s) = \left[\frac{G_1 G_2 G_3}{1 + G_1 G_2 G_3 G_4}\right]$$

## 2.8. Experimental Determination of Unit Impulse Responses and Transfer Functions

If we have the system equations, we can determine the unit impulse responses and transfer functions directly from the equations. Experimental determinations of unit impulse responses and transfer functions are desirable to check the system equations and are necessary when you don't know enough about the process to model it. In the latter case, determination of the unit impulse responses and transfer functions for the process will help you develop a model of the process.

In Section 2.7 we discussed how to approximate a unit impulse with a short rectangular pulse. Note that a unit impulse doesn't have to be made in order to determine a unit impulse response: the response of an output variable in a linear system to an impulse of any magnitude is directly proportional to the response to a unit impulse. However, it is desirable to verify that the does behave linearly by performing experiments with impulses of different magnitudes and check to see that the responses are proportional to each other.

Also note that any unit impulse response and its corresponding transfer function can be determined experimentally using any arbitrary input for which the Laplace transform can be determined, since: (a) the transfer function is equal to the transform of the input function divided by the transform of the output response, and (b) the unit impulse response is the inverse transform of the transfer function. A common experimental input is an approximate positive or negative step. The response to a positive unit step input is the integral over time of the response to a unit impulse. Thus, the derivative with respect to time of the response to a positive unit step input equals the unit impulse response.

For a fluid flow system, the unit impulse response of an inert tracer, g(t)[1/dimension of time], is the same as the "residence time distribution function" of the flow system, E(t)[1/dimension of time]. See H.S. Folger, "Elements of Chemical Reaction Engineering," 2nd ed., Section 13.2, pp. 712-719, for a definition of the residence time distribution function.